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The Dividend Discount Model with Multiple Growth Rates of Any Order for Stock Evaluation

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Abstract

This paper provides a general solution for the dividend discount model in order to compute the intrinsic value of a common stock that allows for multiple stage growth rates of any predetermined number of periods. A mathematical proof is provided for the suggested general solution. A numerical application is also presented. The solution introduced in this paper is expected to improve on the precision of stock valuation, which might be of fundamental importance for the individual investor as well as for financial institutions.

Running title: Stock Evaluation with Multiple Distinct Growth Rates

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1. Introduction

Determining a measure that can represent the intrinsic value of a stock is an important issue for investors and financial institutions seeking profitable investment prospects. This issue might be also of interest for policy makers in order to design appropriate financial guidance. The dividend discount model (DDM), which was originally been developed by Gordon and Shapiro (1956) and Gordon (1959, 1962), can be used for this purpose.³ Currently, there are extensions of the model in the literature that allow for the valuation of a common stock with two different growth rates across time at the maximum, to our best knowledge.⁴ In this paper we suggest a general solution for the valuation of common stocks for the DDM that allows for multiple growth rates of any predetermined number. The suggested solution is proved mathematically and an application is provided.

The remaining part of this paper is organized as follows. The next section presents the model and provides a general solution along with a mathematical proof. Section 3 applies the suggested solution to value a common stock with three different growth rates. The last section offers the concluding remarks.

2. The Model with Multiple Growth Rates for Stock Evaluation

We consider N different growth annuity periods T_1 with growth g_1 , T_2 with growth g_2 , \dots , and T_N with growth g_N , followed by a perpetuity with growth rate g_{N+1} . For $1 \leq l \leq N$, we denote by $ST_l := \sum_{i=0}^{i=l} T_i$ the end of period T_l . The time is to be going from $t = T_0 := 0$ to $ST_N := \sum_{i=0}^{i=N} T_i$ with N annuities and from $ST_N + 1$ to ∞ with a perpetuity as explained in the following:

During $T_0 \rightarrow T_1 = ST_1$ the growth rate is g_1 ,

The dividend at year i before ST_l is $D_i = D_0(1 + g_1)^i$

³ See also Gordon and Gould (1978) as well as Fuller and Hsia (1984).

⁴ It should be mentioned that there are also alternative models for stock valuation such as the capital asset pricing model developed by Sharpe (1964) and the arbitrage pricing theory suggested by Ross (1976).

The dividend at the end of the period is $D_{ST_1} = D_0(1 + g_1)^{T_1}$

During $T_1 + 1 = ST_1 + 1 \rightarrow T_1 + T_2 = ST_2$ the growth rate is g_2 ,

The dividend at year i between ST_1 and ST_2 is $D_i = D_{ST_1}(1 + g_2)^i$

The dividend at the end of the period is $D_{ST_2} = D_{ST_1}(1 + g_2)^{T_2}$

Continuing recursively until the following:

During $ST_{k-1} + 1 \rightarrow ST_k$ the growth rate is g_k ,

The dividend at year i between ST_{k-1} and ST_k is $D_i = D_{ST_{k-1}}(1 + g_k)^i$

The dividend at the end of the period is $D_{ST_k} = D_{ST_{k-1}}(1 + g_k)^{T_k}$

And similarly by continuing we have

During $ST_{N-1} + 1 \rightarrow ST_N$ the growth rate is g_N ,

The dividend at year i between ST_{N-1} and ST_N is $D_i = D_{ST_{N-1}}(1 + g_N)^i$

The dividend at the end of the period is $D_{ST_N} = D_{ST_{N-1}}(1 + g_N)^{T_N}$.

Moreover, for any $1 \leq k \leq N$ and $0 \leq l \leq T_{k+1}$ we can express the following:

$$D_{ST_k+l} = D_{ST_k}(1 + g_{k+1})^l = \left[D_0 \prod_{i=0}^k (1 + g_i)^{T_i} \right] (1 + g_{k+1})^l. \quad (1)$$

The dividend after the $ST_N + m$ years is $D_{ST_N+m} = D_{ST_N}(1 + g_{N+1})^m$.

Now we can present our main result via the following proposition.

Proposition 1 The value of a share, P_0 , is given by

$$P_0 = \sum_{k=1}^{N+1} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k, \quad (2)$$

where

$$A_k := \left[1 - \left(\frac{1+g_k}{1+r} \right)^{T_k} \right], \quad \text{for any } 1 \leq k \leq N, \quad (3)$$

$$A_{N+1} := 1. \quad (4)$$

Proof. First, notice that the value of a share is given -in general terms- by the series

$$P_0 = \sum_{i=1}^{\infty} \frac{D_i}{(1+r)^i}, \quad (5)$$

where D_i denotes the dividend at year i . In the case of N annuities periods T_1, \dots, T_N with respective growth rate g_1, \dots, g_N followed by a perpetuity with growth rate g_{N+1} the value of a share can be written as

$$\begin{aligned} P_0 &= \sum_{i=1}^{T_1} \frac{D_0(1+g_1)^i}{(1+r)^i} + \sum_{i_2=T_1+1}^{T_1+T_2} \frac{D_0(1+g_1)^{T_1}(1+g_2)^{i_2-T_1}}{(1+r)^{i_2}} + \dots \\ &+ \sum_{i_k=ST_{k-1}+1}^{ST_k} \frac{D_{ST_{k-1}}(1+g_k)^{i_k-ST_{k-1}}}{(1+r)^{i_k}} + \dots + \sum_{i_N=ST_{N-1}+1}^{ST_N} \frac{D_{ST_{N-1}}(1+g_N)^{i_N-ST_{N-1}}}{(1+r)^{i_N}} \\ &+ \sum_{i_{N+1}=ST_N+1}^{\infty} \frac{D_{ST_N}(1+g_{N+1})^{i_{N+1}-ST_N}}{(1+r)^{i_{N+1}}}. \end{aligned}$$

Then,

$$\begin{aligned} P_0 &= \sum_{k=1}^N \sum_{i_k=ST_{k-1}+1}^{ST_k} \frac{D_{ST_{k-1}}(1+g_k)^{i_k-ST_{k-1}}}{(1+r)^{i_k}} + \sum_{i_{N+1}=ST_N+1}^{\infty} \frac{D_{ST_N}(1+g_{N+1})^{i_{N+1}-ST_N}}{(1+r)^{i_{N+1}}} \\ &= \sum_{k=1}^N D_{ST_{k-1}} \sum_{l=1}^{ST_k-ST_{k-1}} \frac{(1+g_k)^l}{(1+r)^{l+ST_{k-1}}} + D_{ST_N} \sum_{j=1}^{\infty} \frac{(1+g_{N+1})^j}{(1+r)^{j+ST_N}} \\ &= \sum_{k=1}^N \frac{D_{ST_{k-1}}}{(1+r)^{ST_{k-1}}} \sum_{l=1}^{T_k} \left[\frac{1+g_k}{1+r} \right]^l + \frac{D_{ST_N}}{(1+r)^{ST_N}} \sum_{j=1}^{\infty} \left[\frac{1+g_{N+1}}{1+r} \right]^j. \quad (6) \end{aligned}$$

The last line contains two sums of geometric series that can be calculated as follows

$$\begin{aligned}
\sum_{l=1}^{T_k} \left[\frac{1+g_k}{1+r} \right]^l &= \sum_{l=0}^{T_k} \left[\frac{1+g_k}{1+r} \right]^l - 1 \\
&= \frac{1 - \left(\frac{1+g_k}{1+r} \right)^{T_k+1}}{1 - \frac{1+g_k}{1+r}} - 1 = \frac{1+g_k}{1+r} \left[\frac{1 - \left(\frac{1+g_k}{1+r} \right)^{T_k}}{1 - \frac{1+g_k}{1+r}} \right] = \frac{1+g_k}{r-g_k} \left[1 - \left(\frac{1+g_k}{1+r} \right)^{T_k} \right], \quad (7)
\end{aligned}$$

and

$$\sum_{j=1}^{\infty} \left[\frac{1+g_{N+1}}{1+r} \right]^j = \sum_{j=0}^{\infty} \left[\frac{1+g_{N+1}}{1+r} \right]^j - 1 = \frac{1}{1 - \frac{1+g_{N+1}}{1+r}} - 1 = \frac{1+g_{N+1}}{r-g_{N+1}}. \quad (8)$$

Substituting equations (7) and (8) in (6) we obtain

$$\begin{aligned}
P_0 &= \sum_{k=1}^N \frac{D_{ST_{k-1}}}{(1+r)^{ST_{k-1}}} \frac{1+g_k}{r-g_k} \left[1 - \left(\frac{1+g_k}{1+r} \right)^{T_k} \right] + \frac{D_{ST_N}}{(1+r)^{ST_N}} \frac{1+g_{N+1}}{r-g_{N+1}}, \\
P_0 &= \sum_{k=1}^{N+1} \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k,
\end{aligned}$$

where A_k , for $1 \leq k \leq N+1$, are given by equations (3) and (4). This ends the underlying proof.

In the previous proposition, we provide the value of a share as a function of different growth rates g_i and dividends D_{ST_i} at end of period ST_i , for the interest rate r . We can derive a more explicit formula as a function of D_0 , with different growth rates g_i and the interest rate r as is shown in the following corollary.

Corollary 1 *The value of a share P_0 is given by*

$$P_0 = D_0 \sum_{k=1}^{N+1} \frac{\prod_{i=0}^{k-1} (1+g_i)^{T_i} (1+g_k)}{(1+r)^{ST_{k-1}}(r-g_k)} A_k, \quad (9)$$

where A_k , for $1 \leq k \leq N+1$, are given by equations (3) and (4).

Proof. The proof is straightforward by embedding the value of $D_{ST_{k-1}+1}$ from (1) into the formula (2).

Example 1 In this example, we consider the case of $N = 1$, i.e. when there is one growth rate g_1 at time T and after that a perpetuity with growth rate g_2 . The time is then going from $T_0 = 0$ to $T_1 = T$ with a growth rate g_1 and after T with growth rate g_2 . The formula (9) for this particular case can be written as

$$P_0 = \frac{D_0(1+g_1)}{(r-g_1)} A_1 + \frac{D_0(1+g_1)^{T_1}(1+g_2)}{(1+r)^{ST_1}(r-g_2)} A_2 = \frac{D_1}{(r-g_1)} \left[1 - \left(\frac{1+g_1}{1+r} \right)^T \right] + \frac{D_{T+1}}{(1+r)^T(r-g_2)}.$$

Example 2 In this example, we consider the case of $N = 2$, i.e. we have annuities with growth rates g_1 for time T_1 , g_2 for time T_2 and after that a perpetuity with growth rate g_3 . The time is then going from $T_0 = 0$ to T_1 with a growth rate g_1 , from $T_1 + 1$ to $T_1 + T_2$ with a growth rate g_2 and after $T_1 + T_2$ with the growth rate g_3 . Using formula (2) and equations (3-4) we have

$$\begin{aligned} P_0 &= \sum_{k=1}^3 \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k \\ &= \frac{D_{ST_0+1}}{(1+r)^{ST_0}(r-g_1)} A_1 + \frac{D_{ST_1+1}}{(1+r)^{ST_1}(r-g_2)} A_2 + \frac{D_{ST_2+1}}{(1+r)^{ST_2}(r-g_3)} A_3, \end{aligned}$$

where

$$ST_0 = 0, \quad ST_1 = T_1, \quad \text{and} \quad ST_2 = T_1 + T_2,$$

and

$$A_1 = \left[1 - \left(\frac{1+g_1}{1+r} \right)^{T_1} \right], \quad A_2 = \left[1 - \left(\frac{1+g_2}{1+r} \right)^{T_2} \right], \quad A_3 = 1.$$

Thus, the following can be expressed:

$$P_0 = \frac{D_1}{(r-g_1)} \left[1 - \left(\frac{1+g_1}{1+r} \right)^{T_1} \right] + \frac{D_{T_1+1}}{(r-g_2)} \left[1 - \left(\frac{1+g_2}{1+r} \right)^{T_2} \right] + \frac{D_{T_1+T_2+1}}{(r-g_3)} \left[1 - \left(\frac{1+g_3}{1+r} \right)^{T_3} \right] + \frac{D_{T_1+T_2+T_3+1}}{(r-g_4)} \left[1 - \left(\frac{1+g_4}{1+r} \right)^{\infty} \right]$$

Example 3 In this example, we consider the case of $N = 3$ i.e. we have annuities with growth rates g_1 for time T_1 , g_2 for time T_2 , g_3 for time T_3 and after that a perpetuity with the growth rate g_4 . Again, using formula (2) and equations (3-4) we have

$$\begin{aligned} P_0 &= \sum_{k=1}^4 \frac{D_{ST_{k-1}+1}}{(1+r)^{ST_{k-1}}(r-g_k)} A_k \\ &= \frac{D_{ST_0+1}}{(1+r)^{ST_0}(r-g_1)} A_1 + \frac{D_{ST_1+1}}{(1+r)^{ST_1}(r-g_2)} A_2 \\ &\quad + \frac{D_{ST_2+1}}{(1+r)^{ST_2}(r-g_3)} A_3 + \frac{D_{ST_3+1}}{(1+r)^{ST_3}(r-g_4)} A_4, \end{aligned}$$

where

$$ST_0 = 0, \quad ST_1 = T_1, \quad ST_2 = T_1 + T_2, \quad ST_3 = T_1 + T_2 + T_3$$

and

$$A_1 = \left[1 - \left(\frac{1+g_1}{1+r} \right)^{T_1} \right], \quad A_2 = \left[1 - \left(\frac{1+g_2}{1+r} \right)^{T_2} \right], \quad A_3 = \left[1 - \left(\frac{1+g_3}{1+r} \right)^{T_3} \right], \quad A_4 = 1.$$

Thus, we have

$$\begin{aligned} P_0 &= \frac{D_1}{(r-g_1)} \left[1 - \left(\frac{1+g_1}{1+r} \right)^{T_1} \right] + \frac{D_{T_1+1}}{(r-g_2)} \left[1 - \left(\frac{1+g_2}{1+r} \right)^{T_2} \right] \\ &\quad + \frac{D_{T_1+T_2+1}}{(r-g_3)} \left[1 - \left(\frac{1+g_3}{1+r} \right)^{T_3} \right] + \frac{D_{T_1+T_2+T_3+1}}{(r-g_4)} \left[1 - \left(\frac{1+g_4}{1+r} \right)^{\infty} \right]. \end{aligned}$$

3. An Application

Suppose that a bank has just paid a dividend per share of \$2. Its dividend is expected to grow at 5% during the 3 forthcoming years, it is expected to grow by 7% in years 4 to 7 and after that it is expected to grow at 6% per year in perpetuity. The required rate of return is assumed to be 9%. This information means that the DDM with three growth rates can be used in order to find the intrinsic value of a share for this corporation.

Based on formula (2) we can express the following:

$$P_0 = \frac{D_1}{r - g_1} \left[1 - \frac{(1 + g_1)^{T_1}}{(1 + r)^{T_1}} \right] + \frac{D_{T_1+1}}{r - g_2} \left[1 - \frac{(1 + g_2)^{T_2}}{(1 + r)^{T_2}} \right] + \frac{\left(\frac{D_{T_1+T_2+1}}{r - g_3} \right)}{(1 + r)^{T_1+T_2}}$$

Table 1: The Application

Variable	Value	Variable	Value
D_0	2	r	0.09
D_1	2.1	T_1	3
g_1	0.05	T_2	4
g_2	0.07	D_{T_1+1}	2.47732
g_3	0.06	$D_{T_1+T_2+1}$	3.21691
		P_0	71.05809

Notes: D_i is the dividend in year i and g_j is the growth rate at period j . T_b means break period at time b . P_0 represents the common stock value based on the suggested model.

As it is evident from Table 1, the present value of the underlying common stock is \$71.05809. This can be used as a benchmark price in order to compare it with the real market price for evaluating whether the price is correctly determined or not. If the stock is not fairly priced, regardless if it is overpriced or underpriced, this

information can be used for finding a strategy that results in a gain by choosing an appropriate long or short position in the underlying stock.

4. Conclusions

Stock valuation is an important issue in financial markets for financial actors. This seems to be the case for both domestic as well as the foreign ones due to increasing globalization. One model that is regularly used for this purpose is the dividend discount model (DDM). Currently, this model allows for only two different growth rates to the best of our knowledge. In this paper we suggest a general solution for the DDM that can handle multiple growth rates of any potential number. The suggested solution is proved mathematically and a numerical application is also provided for three different growth rates. The closed form solution introduced in this paper is expected to improve on the precision of stock valuation for different economic conditions, which might be of fundamental importance for investors, financial institutions as well as policy makers.

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