**ISSN 2079-7141**

UAEU College of Business and Economics

*UAEU-CBE-Working Paper Series*

Title: **A New Stochastic Volatility Model**

Author(s): Youssef El-Khatib and Abdulnasser Hatemi-J

Department: Economics and Finance

No. 2015-03

# A New Stochastic Volatility Model

Youssef El-Khatib* and Abdulnasser Hatemi-J**

*UAE University, Department of Mathematical Sciences, Al-Ain, P.O. Box 15551. United Arab Emirates, E-mail: Youssef_Elkhatib@uaeu.ac.ae

** UAE University, Department of Economics and Finance, Al-Ain, P.O. Box 15551, United Arab Emirates. E-mail: AHatemi@uaeu.ac.ae

Abstract

Stochastic volatility modelling is of fundamental importance in financial risk management. Among the most popular existing models in the literature are the Heston and the CEV stochastic models. Each of these models has some advantages that the other one lacks. For example, the CEV model and the Heston model have different relative properties concerning the leverage as well as the smile effects. In this work we deal with the hybrid stochastic volatility model that is based on the CEV and the Heston models combined. This alternative model is expected to perform better than any of the two previously mentioned models in terms of dealing with both the leverage and the smile effects. We deal with the pricing and hedging problems for European options. We first find the set of equivalent martingale measures (E.M.M.). The market is found to be incomplete within this framework since there are infinitely many of E.M.M. We then find the targeted E.M.M. by minimizing the entropy. Using Ito calculus and risk-neutral method enable us to find the partial differential equation (P.D.E.) corresponding to the option price. Moreover, we use Clark-Ocone formula to obtain a hedging strategy that minimizes the distance between the payoff and the value of the hedged portfolio at the maturity. This hedging strategy is among the most efficient available strategies.

**Keywords**: Asset Pricing and Hedging, Options, Stochastic Volatility Model, CEV Model, Heston Model.
**Mathematics Subject Classification** (2010): 91B25, 91G20, 60H07.
JEL Classification: C02, G01, G11, G12, G13

## 1. Introduction

The classical Black and Scholes model (see [2]) is used regularly for the evaluation of options. However, this model suffers from several deficiencies among other the so called smile effect as well as the leverage effect. A well-known approach that for improving the Black Scholes model is to incorporate jumps in the stochastic process. The literature contains quite large number of research work on this issue, we can cite for instance [12], [8] and [6]. In [7] and [9] we can find new types of stochastic volatility models where the main objective is to try to capture the impact of the financial crises. In [1] the author suggests a model that combines stochastic volatility and jumps. In addition, the stochastic volatility models are considered useful tools for taking into account the smile phenomena and to some extent the leverage effect. One of the most popular stochastic volatility model is the Heston model ([5]). Another useful model within this context is the Constant Elasticity Variance (CEV) model developed by Cox ([3]), which is also widely used by practitioners to capture the leverage effect. This paper suggests a combined[1] Heston-CEV model, which is expected to sustain the advantages of each model while reducing their weaknesses.

The remaining part of the paper is organized as follows. Section 2 presents the model. Section 3 deals with the pricing of European options within this new context together with the underlying hedging strategy. The last section concludes the paper.

## 2. The Model

Assume that the probability space is $(\Omega, F, P)$. Assume also that $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ are two Brownian motion processes such that $d\langle W_t, B_t\rangle = \rho dt$ and $|\rho| < 1$. We also consider the filtration $(F_t)_{t\in[0,T]}$ to be the natural filtration generated by $W$ and $B$. The market is consisting of two assets: a risky asset $S = (S_t)_{t\in[0,T]}$ to which is related an European call option and a riskless one given by

$$dA_t = r_t A_t dt, \quad t \in 0, T], \quad A_0 = 1, \tag{1}$$

[1] The combined Heston-CEV model has independently been investigated by others see for example [10].

where $r_t$ is a deterministic measure of time varying interest rate. Assume that the data generating process for the stock price at time $t$, denoted by $S_t$, is the following stochastic differential equation:

$$dS_t = \mu_t S_t dt + \sigma S_t^{\alpha} \sqrt{Y_t} dW_t, \tag{2}$$

$$dY_t = \nu(\theta - Y_t)dt + b\sqrt{Y_t} dB_t \tag{3}$$

where $t \in [0,T]$ and $S_0 = x > 0$. The parameters $\sigma$, $\alpha$, $\nu$, $\theta$ and $b$ are all constant numbers and $\mu_t$ is a deterministic function. Note that $\sigma$ is related to the volatility of the underlying asset, $\alpha$ is the elasticity of the underlying asset variance.

### 2.1 Change of Probability and Equivalent Martingale Measures

In order to insure the no arbitrage condition and according to the first fundamental theorem of asset pricing we need to move to a new probabilistic environment where the probability is a $P$-Equivalent Martingale Measure ($P$-EMM). It is well-known that if $Q$ is a $P$-equivalent probability then by the Radon-Nikodym theorem there exists a $\Phi_T$-measurable random variable, $\rho_T$ such that $Q(A) = E_P[\rho_T 1_A]$, $A \in \Pi(\Omega)$. Notice that $\rho_T$ is strictly positive $P$-a.s, since $Q$ is equivalent to $P$ and $E_P[\rho_T] = E_P[\rho_T 1_\Omega] = 1$. It is common to use the notation $\rho_T := \dfrac{dQ}{dP}$. Consider now the $P$-martingale $\rho = (\rho_t)_{t\in[0,T]}$ defined by

$$\rho_t := E_P[\rho_T \mid \Phi_t] = E_P\left[\frac{dQ}{dP} \mid \Phi_t\right].$$

The next proposition gives the Radon-Nikodym density of an EMM with respect to $P$. Let $Q$ be a $P$-EMM. The Radon-Nikodym density of $Q$ with respect to $P$ is given by

$$\rho_T = \exp\left( \int_0^T (\beta_t dW_t + \gamma_t dB_t) - \frac{1}{2}\int_0^T (\beta_t^2 + \gamma_t^2 + 2\rho\beta_t\gamma_t)dt \right) \tag{4}$$

where $(\beta_t)_{t\in[0,T]}$ and $(\gamma_t)_{t\in[0,T]}$ are two predictable processes. Moreover $\beta_t$ and $\gamma_t$ are related by

$$\mu_t - r_t + \sigma S_t^{\alpha-1}\sqrt{Y_t}(\beta_t + \rho\gamma_t) = 0. \tag{5}$$

*Proof.* A complete proof is available on request.

The previous proposition leads to the following corollary. The market of the model (1-3) is incomplete.

*Proof.* A complete proof is available on request.

We just saw from the previous proposition that there is an infinite number of $P$-EMM. We find the $P$-EMM that minimizes the relative entropy because this will minimize the Kullback–-Leibler distance within these settings (see for instance [11] and [14]). Our aim is to minimize

$$I(Q^\gamma, P) = E_P\left[ \frac{dQ^\gamma}{dP} \ln \frac{dQ^\gamma}{dP} \right] = E_P\left[\rho_T^\gamma \ln \rho_T^\gamma\right], \tag{6}$$

over all the $P$-EMM. The following proposition gives the $P$-EMM that minimizes the relative entropy. Let $\hat{\gamma} = 0$ and $\hat{\beta} = \frac{r_t - \mu_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}}$. The $P$-EMM $\hat{Q}$ defined by its Radon-Nikodym density

$$e_T = \exp\left( \int_0^T \frac{r_t - \mu_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}} dW_t - \frac{1}{2}\int_0^T \left( \frac{r_t - \mu_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}} \right)^2 dt \right),$$

minimizes the relative entropy.

*Proof.* Since we deal with continuous stochastic processes we can apply theorem 1 of [13] which shows that the reverse relative entropy $I(P,Q^{\gamma}) = E_{Q^{\gamma}}\left[\frac{dP}{dQ^{\gamma}}\ln\frac{dP}{dQ^{\gamma}}\right]$ can be used instead of the relative entropy given by (6). We have

$$\beta_t^2 + 2\rho\beta\gamma_t = (\beta_t + \rho\gamma_t)^2 - \rho\gamma_t^2 = \left(\frac{\mu_t - r_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}}\right)^2 - \rho\gamma_t^2.$$

Thus,

$$I(P,Q^{\gamma}) = E_P\left[\frac{1}{2}\int_0^T\left(\gamma_t^2(1-\rho) + \left(\frac{\mu_t - r_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}}\right)^2\right)dt\right].$$

Therefore, we need to minimize the following function:

$$f(x) = \frac{1}{2}\left((1-\rho)x^2 + \left(\frac{\mu_t - r_t}{\sigma S_t^{\alpha-1}\sqrt{Y_t}}\right)^2\right).$$

Note that since $|\rho| < 1$, $f(x)$ has an absolute minimum at $x = 0$. This ends the proof.

## 3. Pricing and Hedging

In this section we find the PDE of the option price as well as a hedging strategy that minimizes the variance. In a complete market, one is interested in finding a strategy that leads to a portfolio value that is equal to the payoff at maturity. In an incomplete model, this type of strategies are not available, thus the question is which one is the best. The answer will depend on in which sense the strategy is better. Here we define the best strategy to be the one that minimizes the distance between the payoff and the value of underlying portfolio (for more details about this approach one can refer to [4]). From now on, we work with $\hat{Q}$ i.e. the $P$-EMM that is minimizing the entropy given by $\hat{\beta}$ from Proposition 2.1. In the previous section, we found the probability measures that insure the market is arbitrage free. Thus, we need to express our model under the new probability space. For this purpose, we define the following:

$$\hat{W}_t = W_t - \int_0^t \hat{\beta}_s ds = W_t - \int_0^t \frac{r_s - \mu_s}{\sigma S_s^{\alpha-1}\sqrt{Y_s}} ds, \quad t \in [0,T].$$

By using the Girsanov theorem $\hat{W}$ is a $\hat{Q}$-Brownian motion. Moreover, under $\hat{Q}$, $(S_t)_{t\in[0,T]}$ satisfies

$$dS_t = r_t S_t dt + \sigma S_t^{\alpha} \sqrt{Y_t} d\hat{W}_t, \quad t \in [0,T],$$
$$dY_t = \nu(\theta - Y_t)dt + b\sqrt{Y_t} dB_t, \quad t \in [0,T].$$

Next we find the price of the option using the PDE approach.

### 3.1 Option Price PDE

The following proposition gives the PDE of the option price for our model. The price of an European call option with maturity $T$ on a stock with price $(S_t)_{t\in[0,T]}$ defined by the model (1), (2) and (3) and with strike $K$ can be written at maturity as $(C := C(t, S_t, Y_t))_{t\in[0,T]}$ and it satisfies the following PDE:

$$C_t + r_t x C_x + \nu(\theta - y)C_y + \frac{1}{2}\sigma^2 x^{2\alpha} y C_{xx} + \frac{1}{2} b^2 y C_{yy} + b\sigma\rho x^{\alpha} y C_{xy} - r_t C = 0, \tag{7}$$

with the terminal condition $C(T, S_T, Y_T) = h(S_T) := (S_T - K)^+$.

*Proof*. By Itô Lemma, we obtain $dC = L_t dt + \sigma S_t^{\alpha}\sqrt{Y_t}C_x d\hat{W}_t + b\sqrt{Y_t}C_y dB_t$, where

$$L_t := C_t + r_t S_t C_x + \nu(\theta - Y_t)C_y + \frac{1}{2}\sigma^2 S_t^{2\alpha} Y_t C_{xx}. + \frac{1}{2}b^2 Y_t C_{yy} + b\sigma\rho S_t^{\alpha} Y_t C_{xy}.$$

Since, $\left(e^{-\int_0^t r_s ds} C\right)_{t\in[0,T]}$ is a $\hat{Q}$-martingale then $L_t = r_t C$ which gives (7). Complete proof is available on request.

In the next subsection we deal with the hedging problem.

### 3.2 Hedging

Let $\eta_t$ and $\zeta_t$ denote the number of units invested at time $t$ in the risky and risk-less assets respectively. Thus the value $V_t$ of the portfolio at time $t$ is given by

$$V_t = \zeta_t A_t + \eta_t S_t, \quad t \in [0,T].$$

Assuming that the portfolio is self-financing, we can state the following. The payoff $h(S_T) = (S_T - K)^+$ is not marketable (attainable). However if $D^{\hat{W}}$ and $D^B$ are the Malliavin derivatives with respect to $\hat{W}$ and $B$ respectively. Then, we have

$$\begin{aligned}
C_0 &= E_{\hat{Q}}[(S_T - K)^+]e^{-\int_0^T r_s ds}, \\
C_x &= \sigma^{-1} S_t^{-\alpha} Y_t^{-\frac{1}{2}} E_{\hat{Q}}[D_t^{\hat{W}}(S_T - K)^+ \mid F_t]e^{-\int_t^T r_s ds}, \qquad (8)\\
C_y &= b^{-1} Y_t^{-\frac{1}{2}} E_{\hat{Q}}[D_t^{B}(S_T - K)^+ \mid F_t]e^{-\int_t^T r_s ds}.
\end{aligned}$$

*Proof.* We use the expansion of $d(e^{-\int_0^t r_s ds} C)$ and the following equalities

$$V_T = V_0 e^{\int_0^T r_t dt} + \int_0^T e^{\int_t^T r_s ds} \eta_t \sigma S_t^{\alpha} \sqrt{Y_t} d\hat{W}_t, \tag{9}$$

$$h(S_T) = E_{\hat{Q}}\left[h(S_T)\right] + \int_0^T E_{\hat{Q}}[D_t^{\hat{W}} h(S_T) \mid F_t] d\hat{W}_t + E_{\hat{Q}}[D_t^B h(S_T) \mid F_t] dB_t. \tag{10}$$

A more detailed proof is available on request from the authors.

The previous proposition is in alignment with the market incompleteness. Since the payoff is not attainable, we search in this case for a portfolio that leads to a value that is the closest to $h(S_T)$. We need to determine in which sense the closeness should be defined. In this paper, we choose to find the hedging strategy that leads to a portfolio that minimizes the distance between the value of the portfolio at maturity $V_T$ and the payoff $h(S_T)$. The next proposition gives the strategy that minimizes the variance $E_{\hat{Q}}\left[(h(S_T) - V_T)^2\right]$. The strategy minimizing $E_{\hat{Q}}\left[(h(S_T) - V_T)^2\right]$ is given by

$$\hat{\eta}_t = \sigma^{-1} S_t^{-\alpha} Y_t^{-\frac{1}{2}} E[D_t^{\hat{W}} (S_T - K)^+ \mid F_t] e^{-\int_t^T r_s ds} = C_x. \tag{11}$$

Moreover, the distance between the payoff and value of the portfolio at maturity is in this case given by the following equation:

$$E_{\hat{Q}}\left[(h(S_T) - \hat{V}_T)^2\right] = \int_0^T (E_{\hat{Q}}[D_t^B h(S_T) \mid \hat{\Phi}_t])^2 dt.$$

*Proof.* By comparing equations (9) and (10) we obtain the following expression:

$$E_{\hat{Q}}\left[(h(S_T)-\hat{V}_T)^2\right]=E_{\hat{Q}}\left[\left(\int_0^T E_{\hat{Q}}[D_t^B f(S_T)\,|\,\hat{\Phi}_t]dB_t\right)^2\right]$$

$$+E_{\hat{Q}}\left[\left(\int_0^T\left(E[D_t^{\hat{W}} f(S_T)\,|\,\hat{\Phi}_t]-e^{\int_t^T r_s ds}\hat{\eta}_t\sigma S_t^{\alpha}\sqrt{Y_t}\right)d\hat{W}_t\right)^2\right]$$

$$=E_{\hat{Q}}\left[\int_0^T g(\hat{\eta}_t)dt\right],$$

where

$$g(x)=(E_{\hat{Q}}[D_t^B f(S_T)\,|\,\hat{\Phi}_t])^2+\left(E_{\hat{Q}}[D_t^{\hat{W}} f(S_T)\,|\,\hat{\Phi}_t]-e^{\int_t^T r_s ds}x\sigma S_t^{\alpha}\sqrt{Y_t}\right)^2.$$

The minimum is reached at $g'(x)=0$. Therefore, the strategy that minimizes the underlying variance is given by equation (11). The second part of the equality is obtained from equation (8). This ends the proof.

## 4. Conclusions

It is widely agreed in the literature that stochastic volatility models are useful tools for risk management in financial markets. This topic is increasingly capturing the focus of researchers in mathematical finance. In this work, an alternative stochastic volatility model has been introduced. It combines the CEV and the Heston models. This combined model is more consistent with the reality than the CEV or the Heston model separately. The pricing and hedging problems for the considered model have been investigated. After providing the Radon-Nikodym density for an arbitrary equivalent martingale measure, we show that the market is incomplete in this scenario. Within this framework, the PDE of the option price for a European call option was derived under the minimal entropy martingale measure. Using the Malliaivn calculus and the Clarck-Ocone formula, the strategy that minimizes the variance was also obtained. A mathematical proof for each proposition is provided. The suggested model can be useful to investors in their continuous pursue for finding up to date strategies that can result in more efficient financial risk management compared to the existing approaches.


## Acknowledgements

A previous version of this paper has been presented at the ECMI 2014—the 18th European Conference on Mathematics for Industry, June 9-13, 2014, in Taormina, Italy. The paper was also presented at the seminar series, at the department of economics and finance, the UAE university in 2014. The authors would like to thank the participants of each presentation for their comments. However, the usual disclaimer applies. Furthermore, the second author would like to acknowledge thankfully the funding received from the UAE national fund via the UPAR-2013 program of the UAE university.

**References**

[1] Bates, D. S.: Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, **9**, 69–107 (1996).

[2] Black F.; Scholes, M.: The pricing of options and corporate liabilities. *Journal of Political Economy*, **81(3)**, 637–654 (1973).

[3] Cox, J.: Notes on option pricing I: constant elasticity of diffusions. Unpublished draft, Stanford University, (1975).

[4] Föllmer, H.; Sondermann, D.: Hedging of non-redundant contingent claims. In: Hildenbrand, W., Mas-Colell, A. (eds.) Contributions to Mathematical Economics, pp. 205-223. North-Holland (1986).

[5] Heston, S.: A closed-form solutions for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, **6**, 327–343 (1993).

[6] El-Khatib, Y., Al-Mdallal, Q.M.: Numerical simulations for the pricing of options in jump diffusion markets. *Arab Journal of Mathematical Sciences* **18(2)**, 199–208 (2012).

[7] El-Khatib, Y., Hatemi-J, A.: Computations of price sensitivities after a financial market crash. *Electrical Engineering and Intelligent Systems*, Lecture Notes in Electrical Engineering, Volume 130, 239–248, (2013).

[8] El-Khatib, Y., Hatemi-J, A.: On the calculation of price sensitivities with a jump-diffusion structure. *Journal of Statistics Applications & Probability*, **1(3)**, 171–182, (2012).

[9] El-Khatib, Y., Hajji, M.A., Al-Refai, M.: Options Pricing in Jump Diffusion Markets during Financial Crisis. *Applied Mathematics & Information Sciences*, **7(6)** (2013).

[10] Forde, M. and Pogudin, A.: The large maturity smile for the SABR and CEV-Heston model. *IJTAF*, **16(8),** (2014).

[11] Frittelli, M.: The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, **10(1),** (2000).

[12] Merton, R. C.: Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3**, 125-44, ( 1976).

[13] Schweizer , M. : A minimality property of the minimal martingale measure, *Statistics and Probability Letters*, **42**, 27–31, (1999).

[14] Schweizer M.: On the minimal martingale measure and the Föllmer-Schweizer decomposition, *Stochastic Analysis and Applications*, **13**, 573–599, (1995).