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Interval Estimation of Variance Ratio in Non-Normal Unbalanced One-Way Random Models

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Abstract

Confidence intervals based on the harmonic mean method are proposed for estimating the ratio of variance components and/or the intraclass correlation coefficient (ρ) in unbalanced one-way random models without the normality assumptions. Since the proposed procedure is heavily dependent on the estimation of the kurtosis of the underlying distributions, bias-corrected estimators of kurtosis are proposed as well. Several asymptotic results concerning the proposed procedure are given along with simulation results to assess its performance in finite sample size situations. The proposed intervals are also compared with the corresponding confidence intervals based on the arithmetic mean method and were found to effectively maintain the nominal probability of coverage, except for leptokurtic distributions with fairly large kurtosis where the intervals tend to be liberal. According to the simulation results, the proposed harmonic mean intervals are recommended for use in practice. However, while computing the harmonic mean intervals, the bias-corrected estimators of kurtosis should be used when it is anticipated that ρ is small, but for large ρ the empirically corrected estimators of kurtosis should be adopted. The procedure is illustrated using a real data set.

Keywords: Interval estimation; Intraclass correlation; Harmonic mean method;

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Kurtosis estimation; Unbalanced mixed models; Variance components.

1. Introduction

Consider the model

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad \text{for } i = 1, \ldots, a \text{ and } j = 1, \ldots, n_i. \tag{1.1}$$

Let $n = \sum_{i=1}^{a} n_i$. It is assumed that $n > a$, which implies that there is at least one replication in the experiment, that α_i and ϵ_{ij} are independent, that α_i is a sequence of independent mean 0 and variance σ_a^2 random variables and that ϵ_{ij} is a sequence of independent mean 0 variance σ_e^2 random variables. Let $\kappa_a = E(\alpha_i^4) - 3\sigma_a^4$ denote the kurtosis of the distribution of α_i and $\kappa_e = E(\epsilon_{ij}^4) - 3\sigma_e^4$ denote the kurtosis of the distribution of ϵ_{ij}. We also denote the standardized kurtosis of α_i and ϵ_{ij} by $\gamma_a = \kappa_a/\sigma_a^4$ and $\gamma_e = \kappa_e/\sigma_e^4$ respectively.

Model (1.1) can be generalized to more complex designs and has proven useful to practitioners in a variety of fields, e.g., see Bonett [2002] for its adaptation to two-way random and mixed models and Cochran [1977] for its applications in subsampling, also referred to as two-stage sampling with units of unequal size. Often the investigator in these applications is interested in estimating the ratio of variance components, $\theta = \sigma_a^2/\sigma_e^2$, and/or the intraclass correlation coefficient, $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$.

Under the assumption that both α_i and ϵ_{ij} are normally distributed, Wald [1940] proposed a procedure based on a pivotal quantity involving the eigenvalues of the variance-covariance matrix of a linear transformation of the observations to construct an exact confidence interval for ρ. As such procedure requires the solution of two nonlinear equations, Harville and Fenech [1985] approximated the eigenvalues by their arithmetic and harmonic means to simplify the computation of the confidence intervals. Burdick et al. [1986] compared Wald's intervals with the approximate intervals based on these two approximations. Their simulation results showed that the arithmetic mean intervals maintained the nominal confidence coefficients for small values of ρ, while the harmonic mean intervals maintained the nominal confidence coefficients for large values of ρ. It should be mentioned that all three methods are identical in the balanced case and that the harmonic mean approximation was first suggested for the unbalanced random one-way model by Thomas and Hultquist [1978]. In addition, controlled widths and coverage confidence intervals based on the pivotal quantity method were also investigated by Zoubeidi and El-Bassiouni [2007], and El-Bassiouni and Zoubeidi [2008].

Harville [1977] considered the maximum likelihood (ML) estimators and the restricted maximum likelihood (REML) estimators of variance components. However, since the computation of the ML and REML estimates requires the numerical solution of a constrained nonlinear optimization problem, ML- and REML-based interval estimation of θ and ρ has not been used much in practice. This led Burch and Harris [2001] to develop a closed-form approximations to the REML estimator of a variance ratio. Further, Jiang [1996, 2005] studied the large-sample properties of the REML estimators, which can be used to construct asymptotic confidence intervals for θ and ρ.

On the other hand, Hannig et al. [2006], Lidong et al. [2008], and Hannig [2009] developed fiducial intervals for variance components in unbalanced normal mixed linear models. The reader is referred to Burdick et al. [2006] for an account of the current status of interval estimation in one-way random models.

Burch [2011*a*] considered interval estimation of θ and ρ in balanced one-way random models without assuming normality and extended this work to unbalanced one-way random models in Burch [2011*b*] where he compared three techniques corresponding to the pivotal quantity, fiducial and REML (based on the arithmetic mean approximation) methods. For non-normal populations, the development of the REML technique made use of the results of Jiang [2005] to show that the REML estimator is asymptotically normally distributed and that its asymptotic variance does not depend on the normality assumption, but is remarkably reliant on estimating the kurtosis of the underlying distributions of α_i and ϵ_{ij}. Kurtosis estimation was investigated also by Shoukri et al. [1990], Teuscher et al. [1994], Singh et al. [2002], and An and Ahmed [2008]. Burch [2011*b*] showed by means of a Monte Carlo Simulation study that REML confidence intervals based on the arithmetic mean approximation and Empirically Corrected Kurtosis Estimators (ECKEs) have outperformed those based on the pivotal quantity and fiducial methods.

In the present paper, REML confidence intervals for θ and ρ based on the harmonic mean approximation as well as Bias-Corrected Kurtosis Estimators (BCKEs) of κ_e and κ_a are proposed.

The rest of the paper is organized as follows: the interval estimation methodology is discussed in Section 2, whereas kurtosis estimation is dealt with in Section 3. Since the results in Sections 2 and 3 are asymptotic in nature, the empirical results of a Monte Carlo simulation study are given in Section 4 to assess the performance of the proposed procedures in moderate sample size situations. An example is worked out in Section 5 to illustrate the proposed procedures. The conclusions are provided in Section 6. The asymptotic normality of the harmonic mean approximation to the REML estimator is established in Appendix A for non-normal unbalanced one-way random models.

2. Methodology

In order to construct the confidence interval for θ (and thus ρ), consider the set of independently distributed quadratic forms denoted $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$, which constitute minimal sufficient statistics for the reduced (location invariant) linear model devoid of the parameter μ, see Olsen et al. [1976]. Lamotte [1976] showed under the assumption of normality that

$$\mathcal{Q}_i \sim \frac{\sigma_e^2}{1-\rho}(1+\rho(\lambda_i-1))\chi^2(r_i),$$

where $i = 1, \ldots, m$ and $0 = \lambda_1 < \ldots < \lambda_m$ represent the distinct eigenvalues of the variance-covariance matrix of a linear transformation of the observations having $r_1, \ldots, r_m$ multiplicities, respectively, see Burch [2011*a*] for additional details. It is also known that $r_1 = n - a$ and $\sum_{i=2}^m r_i = a - 1$. One can show that $\mathcal{Q}_1$ represents the sum of squares of errors (within groups)

$$\mathcal{Q}_1 = SSE = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i.})^2$$

and $W_A = \sum_{j=2}^m \mathcal{Q}_j = SSA = \sum_{i=1}^a n_i(\bar{Y}_{i.} - \bar{Y}_{..})^2$ represents the sum of squares among groups, where $\bar{Y}_{i.}$ and $\bar{Y}_{..}$ are respectively the ith group mean and overall mean. Studying the properties of $\mathcal{Q}_1$ and $\sum_{j=2}^m \mathcal{Q}_j$, Wald [1940] constructed a pivotal quantity of θ which allows the development of a confidence interval for the ratio of variance components in a random model with unbalanced data, see also Searle et al. [2006].

2.1. Arithmetic Mean Estimator of θ

Under the assumption of normality, the Restricted Maximum Likelihood (REML) estimators of (σ_e^2, θ) are obtained by maximizing

$$L_R \sim -(n-1)\ln(\sigma_e^2) - \sum_{j=2}^{m} r_j \ln(1+\lambda_j\theta) - \frac{1}{\sigma_e^2}\left(\mathcal{Q}_1 + \sum_{j=2}^{m} \frac{\mathcal{Q}_j}{1+\lambda_j\theta}\right) \quad (2.1)$$

Searle et al. [2006] showed that no closed form expressions for the REML estimators can be obtained. Burch [2011*b*] proposed an approximate restricted log-likelihood function by replacing λ_j by $\bar{\lambda}$, for $j = 2, \ldots, m$, in L_R leading to

$$L_R^A \sim -(n-1)\ln(\sigma_e^2) + (a-1)\ln(1+\bar{\lambda}\theta) - \frac{1}{\sigma_e^2}\left(\mathcal{Q}_1 + \frac{1}{1+\bar{\lambda}\theta}\sum_{j=2}^{m} \mathcal{Q}_j\right),$$

where

$$\bar{\lambda} = \frac{1}{a-1}\sum_{j=2}^{m} r_j\lambda_j = \frac{n-\eta_2}{a-1},$$

with $\eta_k = \frac{1}{n}\sum_{i=1}^{a} n_i^k$ for k integer.

Differentiating L_R^A and solving the likelihood equations yields the following approximate REML estimator of θ

$$\hat{\theta}_A = \frac{1}{\bar{\lambda}}\left(\frac{(n-a)W_A}{(a-1)\mathcal{Q}_1} - 1\right) = \frac{1}{\bar{\lambda}}\left(\frac{MSA}{MSE} - 1\right),$$

if $MSA = \frac{SSA}{a-1} \geq MSE = \frac{SSE}{n-a}$ and zero otherwise. This estimator can also be seen as an arithmetic mean (of the λ_i's) approximation of the estimator proposed by Wald [1940]. To study the asymptotic of $\hat{\theta}_A$, one first recall the following computations from Hammersley [1949],

$$\begin{aligned}
\mathrm{E}(\mathcal{Q}_1) &= (n-a)\sigma_e^2, \\
\mathrm{Var}(\mathcal{Q}_1) &= 2(n-a)\sigma_e^4 + \kappa_e(n\eta_{-1} + n - 2a), \\
\mathrm{E}(W_A) &= (a-1)\sigma_e^2 + (n-\eta_2)\sigma_a^2, \\
\mathrm{Var}(W_A) &= 2A_1\sigma_a^4 + A_2\kappa_a + 2A_3\sigma_e^4 + A_4\kappa_e + 4A_5\sigma_a^2\sigma_e^2, \\
\mathrm{Cov}(W_A, \mathcal{Q}_1) &= (a-1+a/n-n\eta_{-1})\kappa_e,
\end{aligned}$$

where

$$
\begin{aligned}
A_1 &= n\eta_2 - 2\eta_3 + \eta_2^2, \\
A_2 &= n\eta_2 - 2\eta_3 + \eta_4/n, \\
A_3 &= a - 1, \\
A_4 &= n\eta_{-1} + (1 - 2a)/n, \\
A_5 &= n - \eta_2.
\end{aligned}
$$

The variance of $\hat{\theta}_A$ is given by $\mathrm{Var}(\hat{\theta}_A) = (n-a)^2 \mathrm{Var}(W_A/\mathcal{Q}_1)/\{\bar{\lambda}^2(a-1)^2\}$ where the variance of $W_A/\mathcal{Q}_1$ is approximated via a linearization argument by

$$
\mathrm{Var}\,(W_A/\mathcal{Q}_1) \simeq \frac{\mathrm{Var}(W_A)\mathrm{E}(\mathcal{Q}_1)^2 + \mathrm{Var}(\mathcal{Q}_1)\mathrm{E}(W_A)^2 - 2\mathrm{Cov}(W_A, \mathcal{Q}_1)\mathrm{E}(W_A)\mathrm{E}(\mathcal{Q}_1)}{\mathrm{E}(\mathcal{Q}_1)^4}. \quad (2.2)
$$

The asymptotic normality of $\hat{\theta}_A$ is established in Burch [2011*b*] where it was noted that, in practice, the distribution of $\hat{\theta}_A$ is right skewed and it is more appropriate to work with the variance stabilizing transformation $\ln(1+\bar{\lambda}\hat{\theta}_A))$, see for instance Bonett [2006]. Using such transformation, Burch [2011*b*] proposed the following confidence interval for θ

$$
\left[\frac{(1+\bar{\lambda}\hat{\theta}_A)e^{-Z_{\alpha/2}\sqrt{\mathrm{Var}(\ln(1+\bar{\lambda}\hat{\theta}_A))}} - 1}{\bar{\lambda}} ; \frac{(1+\bar{\lambda}\hat{\theta}_A)e^{Z_{\alpha/2}\sqrt{\mathrm{Var}(\ln(1+\bar{\lambda}\hat{\theta}_A))}} - 1}{\bar{\lambda}} \right],
$$

where $Z_{\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. The variance of $\ln(1+\bar{\lambda}\hat{\theta}_A))$, is easily approximated by

$$
\mathrm{Var}(\ln(1+\bar{\lambda}\hat{\theta}_A)) \simeq \frac{\bar{\lambda}^2}{(1+\bar{\lambda}\hat{\theta}_A)^2}\mathrm{Var}(\hat{\theta}_A).
$$

2.2. Harmonic Mean Estimator of θ

As shown by Thomas and Hultquist [1978] a better estimator, in the case of large θ, could be obtained if the λ_j are replaced by their harmonic mean $\bar{\lambda}_H = (a-1)/\sum_{j=2}^m r_j/\lambda_j = a/(n\eta_{-1})$ instead of their arithmetic mean $\bar{\lambda}$. Applying this approach to the likelihood function (2.1) yields

$$
L_R^H \sim -(n-1)\ln(\sigma_e^2) - (a-1)\ln(1+\bar{\lambda}_H\theta) - \frac{1}{\sigma_e^2}\left(\mathcal{Q}_1 + \frac{\bar{\lambda}_H}{1+\bar{\lambda}_H\theta}\sum_{j=2}^m \mathcal{Q}_j/\lambda_j \right).
$$

First set $W_H = \sum_{j=2}^{m} \mathcal{Q}_j/\lambda_j = SSU = \sum_{i=1}^{a}(\bar{Y}_{i.} - \bar{Y}_H)^2$ where $\bar{Y}_H = \sum_{i=1}^{a}\bar{Y}_{i.}/a$ and let $MSU = SSU/(a-1)$. Next, differentiating L_R^H and solving the likelihood equations provides the following estimator of θ

$$\hat{\theta}_H = \frac{(n-a)W_H}{(a-1)\mathcal{Q}_1} - \frac{1}{\bar{\lambda}_H} = \frac{MSU}{MSE} - \frac{1}{\bar{\lambda}_H},$$

if $MSU/MSE \geq 1/\bar{\lambda}_H$ and zero otherwise. Note that SSU and MSU represent the unweighted sum and mean of squares among groups, respectively, and $\bar{\lambda}_H$ is the harmonic mean of the group sizes (El-Bassiouni and Zoubeidi [2008]).

Direct computation gives

$$\begin{array}{lcl} \mathrm{E}(W_H) & = & (a-1)\sigma_a^2 + \{n\eta_{-1}(a-1)/a\}\sigma_e^2, \\ \mathrm{Var}(W_H) & = & 2H_1\sigma_a^4 + H_2\kappa_a + 2H_3\sigma_e^4 + H_4\kappa_e + 4H_5\sigma_a^2\sigma_e^2, \\ \mathrm{Cov}(W_H, \mathcal{Q}_1) & = & \kappa_e n(a-1)(\eta_{-1} - \eta_{-2})/a, \end{array}$$

where

$$\begin{array}{lcl} H_1 & = & a-1 \\ H_2 & = & (a-1)^2/a \\ H_3 & = & n\,(\eta_{-2}a^2 - 2\,a\eta_{-2} + n\eta_{-1}{}^2)/a^2 \\ H_4 & = & n\eta_{-3}\,(a-1)^2/a^2 \\ H_5 & = & n\eta_{-1}\,(a-1)\,/a. \end{array}$$

One then easily computes $\mathrm{Var}(\hat{\theta}_H) = (n-a)^2\mathrm{Var}(W_H/\mathcal{Q}_1)/(a-1)^2$, where $\mathrm{Var}(W_H/\mathcal{Q}_1)$ is approximated using Formula (2.2) with W_A replaced by W_H. The asymptotic normality of $\hat{\theta}_H$ is established in Appendix A. However, for finite samples, the distribution of $\hat{\theta}_H$ is right skewed and the transformation $\ln(1 + \bar{\lambda}_H\hat{\theta}_H))$ reduces such skewness and stabilizes the variance. As in the previous section, a confidence interval for θ is then given by

$$\left[\frac{(1+\bar{\lambda}_H\hat{\theta}_H)e^{-Z_{\alpha/2}\sqrt{\mathrm{Var}(\ln(1+\bar{\lambda}_H\hat{\theta}_H))}} - 1}{\bar{\lambda}_H}; \frac{(1+\bar{\lambda}_H\hat{\theta}_H)e^{Z_{\alpha/2}\sqrt{\mathrm{Var}(\ln(1+\bar{\lambda}_H\hat{\theta}_H))}} - 1}{\bar{\lambda}_H}\right],$$

where $\mathrm{Var}(\ln(1+\bar{\lambda}_H\hat{\theta}_H)) \simeq \bar{\lambda}_H^2\mathrm{Var}(\hat{\theta}_H)/(1+\bar{\lambda}_H\hat{\theta}_H)^2$.

It should be noted that the computation of the arithmetic and harmonic mean approximations to the REML does not require the computation of any eigenvalues or eigenvectors since $\hat{\theta}_A$ and $\hat{\theta}_H$ are defined in terms of the mean squares readily available in weighted and unweighted analyses of variance. As well, $\bar{\lambda}$ and $\bar{\lambda}_H$ are defined in terms of the group sizes.

3. Kurtosis Estimation

A close look at the expressions of the variances given above indicates that they all depend on κ_e and κ_a. Estimators of these measures of kurtosis are therefore required in order to compute the proposed confidence intervals. The estimation of κ_e is discussed first.

The most intuitive estimator for κ_e is given by

$$\hat{\kappa}_e = \frac{\sum_{i=1}^{a} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^4}{n} - 3 \left[\frac{\sum_{i=1}^{a} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2}{n} \right]^2 .$$

Unfortunately such estimator is biased and its expectation is given by

$$E(\hat{\kappa}_e) \quad = \quad \frac{\sum_{i=1}^{a} n_i E(Y_{ij} - \bar{Y}_{i.})^4}{n} - 3 \frac{E(SSE^2)}{n^2} = D_1 \kappa_e + D_2 \sigma_e^4,$$

where

$$\begin{aligned} D_1 &= \frac{n^2 - 4\,an - 3\,n^2 \eta_{-2} + 6\,n^2 \eta_{-1} - 3\,n\eta_{-1} - 3\,n + 6\,a}{n^2} \\ D_2 &= \frac{3n^2 \eta_{-1} - 6n + 6a - 3a^2}{n^2}. \end{aligned}$$

Letting $\hat{\kappa}_e^* = \hat{\kappa}_e / D_1$ one obtains an estimator of κ_e with bias given by $E(\hat{\kappa}_e^*) - \kappa_e = D_2 \sigma_e^4 / D_1$. Plugging an estimator of σ_e^2 like the MSE yields the following BCKE of κ_e

$$\hat{\kappa}_e^c = \frac{\hat{\kappa}_e - D_2 \mathrm{MSE}^2}{D_1}. \tag{3.1}$$

Going along similar lines, an intuitive estimator of κ_a, the kurtosis of α, is given by

$$\hat{\kappa}_a = \frac{\sum_{i=1}^{a} n_i (\bar{Y}_{i.} - \bar{Y}_{..})^4}{n} - 3 \left[\frac{\sum_{i=1}^{a} n_i (\bar{Y}_{i.} - \bar{Y}_{..})^2}{n} \right]^2 .$$

The above estimator is also biased and its expectation is given by

$$E(\hat{\kappa}_a) = \frac{\sum_{i=1}^{a} n_i E\left[(\bar{Y}_{i.} - \bar{Y}_{..})^4 \right]}{n} - \frac{3}{n^2} E(SSA^2),$$

which after long but straightforward computation reduces to

$$E(\hat{\kappa}_a) = C_1\kappa_a + C_2\sigma_a^4 + C_3\kappa_e + C_4\sigma_e^4 + C_5\sigma_a^2\sigma_e^2,$$

where

$$\begin{aligned}
C_1 &= (n^3 - 7n^2\eta_2 + 12n\eta_3 - 6\eta_4)/n^3 \\
C_2 &= -6(n\eta_2 + 3\eta_2^2 - 4\eta_3)/n^2 \\
C_3 &= (-6 + 12a + n^3\eta_{-2} - 7n^2\eta_{-1})/n^3 \\
C_4 &= -3(-n^2\eta_{-1} - 2 + 2a + a^2)/n^2 \\
C_5 &= 12(\eta_2 - 2n + a\eta_2)/n^2.
\end{aligned}$$

In an analogous manner to that followed in the case of κ_e, a bias-corrected estimator for κ_a is obtained by replacing σ_e^2, κ_e and θ by their estimators and then dividing by C_1 leading to

$$\hat{\kappa}_a^c = \frac{\hat{\kappa}_a - (C_2\hat{\theta}^2\text{MSE}^2 + C_3\hat{\kappa}_e^c + C_4\text{MSE}^2 + C_5\hat{\theta}\text{MSE}^2)}{C_1}. \qquad (3.2)$$

A quite similar bias correction can also be found in Teuscher et al. [1994] who used $\sum_{i=1}^a \sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i.})^4$ and $\sum_{i=1}^a n_i(\bar{Y}_{i.} - \bar{Y}_{..})^4$ and applied a similar procedure to the above to obtain bias-corrected estimators for κ_e and κ_a. Teuscher et al. [1994] also applied a similar procedure to $\sum_{i=1}^a \sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_{i.})^3$ and $\sum_{i=1}^a n_i(\bar{Y}_{i.}-\bar{Y}_{..})^3$ to construct bias-corrected estimators for the skewness of the distributions of ϵ_{ij} and α_i. They also proposed further corrections of these estimators by projecting them on the space of feasible values of skewness and kurtosis.

Burch [2011*b*] proposed an alternative bias correction which focuses on the estimation of $\gamma_a^* = \kappa_a/\sigma_e^4 = \gamma_a\theta^2$ and $\gamma_e = \kappa_e/\sigma_e^4$. He first used the statistics $\hat{\kappa}_a/\text{MSE}^2$ and $\hat{\kappa}_e/\text{MSE}^2$ and computed their expectation under the normality assumption. He then calculated $\bar{\gamma}_a^* = \hat{\kappa}_a/\text{MSE}^2 - E_N(\hat{\kappa}_a/\text{MSE}^2)$ and $\bar{\gamma}_e = \hat{\kappa}_e/\text{MSE}^2 - E_N(\hat{\kappa}_e/\text{MSE}^2)$, where E_N denotes the expectation under the normality assumption. To further reduce the bias, Burch [2011*b*] proposed to use $\bar{\gamma}_a^{*e} = g_1(\bar{\gamma}_a^*)$ and $\bar{\gamma}_e^e = g_2(\bar{\gamma}_e)$ where g_1 and g_2 are determined empirically via a simulation study. In Burch [2011*b*], it is suggested to use $g_1(x) = 0.1x$ and $g_2(x) = 2x + 2x^2$.

4. Simulation Study

The finite sample properties of the above estimators are investigated via a simulation study involving different scenarios. Both coverage probabilities and the expected lengths of the proposed confidence intervals for θ are compared. For simplicity, these confidence intervals are converted into confidence intervals for the intraclass correlation coefficient $\rho = \theta/(1+\theta)$. For the sake of comparison with Burch [2011*b*], the same simulation setup is adopted. Precisely, the four designs listed in Table 1 are used with the number of groups $a \in \{5, 10, 50, 100\}$. For $a = 5$, each design has a total sample size of $n = \sum_{i=1}^{a} n_i = 24$. For the other values of a, the designs in Table 1 are simply replicated. Thus, for $a = 10$, 50 and 100, the sample sizes are respectively 48, 240 and 480. To measure the imbalance of each design, we

Table 1: Group size patterns for $a = 5$

Pattern	n_1	n_2	n_3	n_4	n_5	Φ
1	5	5	5	5	4	0.99
2	10	5	5	2	2	0.69
3	10	10	2	1	1	0.39
4	20	1	1	1	1	0.26

computed $\Phi = (\frac{a}{n})(\frac{a}{n\eta_{-1}})$ as defined by Ahrens and Pincus [1981]. Note that $0 \leq \Phi \leq 1$, and Φ is equal to one if the model is balanced. Without loss of generality, we let $\mu = 0$ and $\sigma_e^2 = 1$. In the simulation, the values of the intraclass correlation coefficient used are $\rho = 0.1, 0.2, \ldots, 0.9$. Further, it was assumed that ϵ_{ij} and α_i are mutually independent and follow a centered and rescaled version of the distributions listed in Table 2. The distribution of ϵ_{ij} is rescaled to have a variance of 1 while that of α_i is rescaled to have a variance $\sigma_a^2 = \rho/(1-\rho)$. Since the estimation of the variances of $\hat{\theta}_A$ and $\hat{\theta}_H$ is directly affected by the estimation of the kurtosis of the underlying distribution, the choice of these distributions provides a reasonable range of possible kurtosis values. The beta and uniform distributions have a negative kurtosis, the normal distribution has a null kurtosis while remaining distributions have a positive kurtosis with the *Exponential*(1), $t(5)$ and $\chi^2(1)$ distributions having a large positive kurtosis. Note that it was assumed throughout the simulation that α_i and ϵ_{ij} have the same underlying distribution.

The coverage probabilities and expected lengths of the nominal 95% confidence intervals for ρ were computed based on 10000 Monte Carlo iterations. The kurtosis is estimated using the BCKEs discussed in Section 3 and the ECKEs of Burch [2011*b*]. In the results, a superscript c is used if the kurtosis is estimated using formulas (3.1) and (3.2) and a superscript e is used if the kurtosis is estimated using the empirical correction of Burch [2011*b*].

Table 2: Distributions and their standardized kurtosis

Distribution	γ_a
Beta(0.4,0.6)	-1.33
Uniform(0,1)	-1.2
N(0,1)	0.0
$t(10)$	1.0
Gamma(2,1)	3.0
Exponential(1)	6.0
$t(5)$	6.0
$\chi^2(1)$	12.0

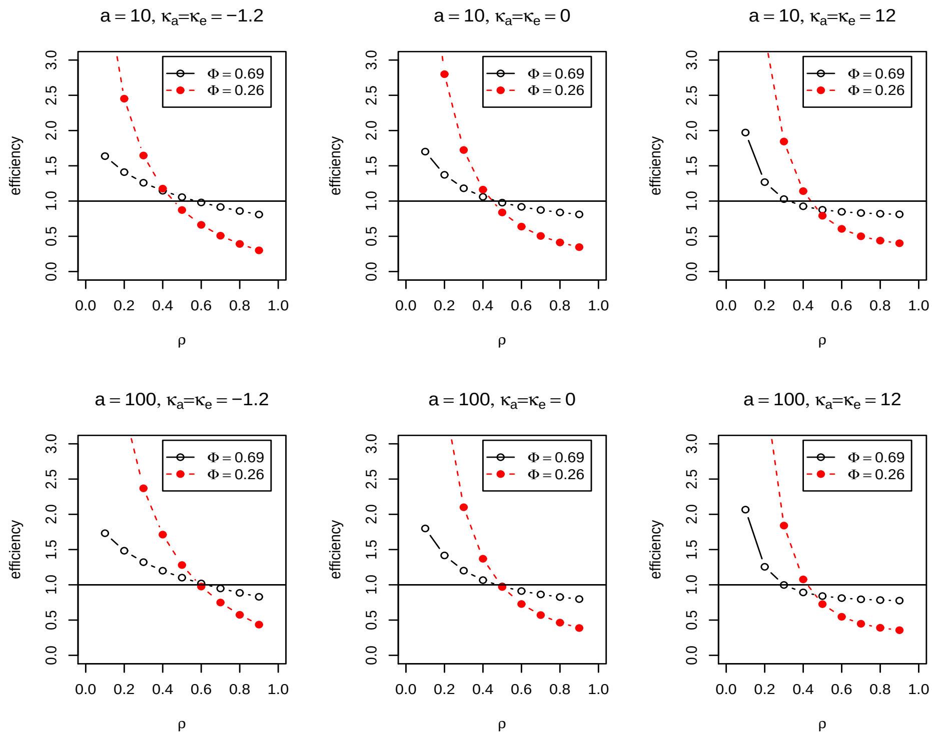
4.1. Theoretical efficiencies of the estimators

Figure 1 shows the ratio of the variance of $\hat{\theta}_H$ to that of $\hat{\theta}_A$ for different designs (degrees of imbalance), different sample sizes and different distributions (measures of kurtosis) for α_i and ϵ_{ij}. In general, the harmonic mean estimator $\hat{\theta}_H$ is more efficient than $\hat{\theta}_A$ for medium and large values of ρ for all distributions and scenarios. The gain in efficiency is more pronounced for unbalanced designs. The ratio of the two variances reaches values less than 50% for large values of ρ and for $\Phi = 0.26$. Though not reported in Figure 1, for nearly balanced patterns ($\Phi \geq 0.9$), the relative efficiencies varied between 0.992 and 1.092, and the two estimators are quite similar.

4.2. Kurtosis estimation

The estimation of the variances of the proposed estimators depends mainly on the estimation of the kurtosis of the underlying distributions of α_i and ϵ_{ij}. Two classes of kurtosis estimators are considered here. The BCKEs denoted by $\hat{\kappa}_a^c$ and $\hat{\kappa}_e^c$, which are based on the formulas (3.2) and (3.1) shown in Section 3, and the ECKEs denoted by $\hat{\kappa}_a^e$ and $\hat{\kappa}_e^e$, which are proposed by Burch [2011*b*]. Figure 2 shows boxplots for the estimators $\hat{\kappa_a}^c$ and $\hat{\kappa_a}^e$ under the $\mathcal{U}(0,1)$, and $\chi^2(1)$ distributions, for the sample size $a = 100$, for $\rho = 0.1$ or 0.9 and for $\Phi = 0.26$. For small values of ρ, the empirical estimator $\hat{\kappa}_a^e$ of the kurtosis is much closer to the true value than the bias-corrected estimator under both distributions. In contrast, for large values of ρ, $\hat{\kappa}_a^c$ provides a better approximation to the true value of κ_a for both distributions. In this case, $\hat{\kappa}_a^e$ overestimate the true value when the distribution is uniform and severely underestimate the true value when the distribution is $\chi^2(1)$. For other distributions and other values of Φ the behavior of $\hat{\kappa}_a^c$ and $\hat{\kappa}_a^e$ are along

Figure 1: Relative efficiencies of $\hat{\theta}_H$ with respect to $\hat{\theta}_A$ under different scenarios.



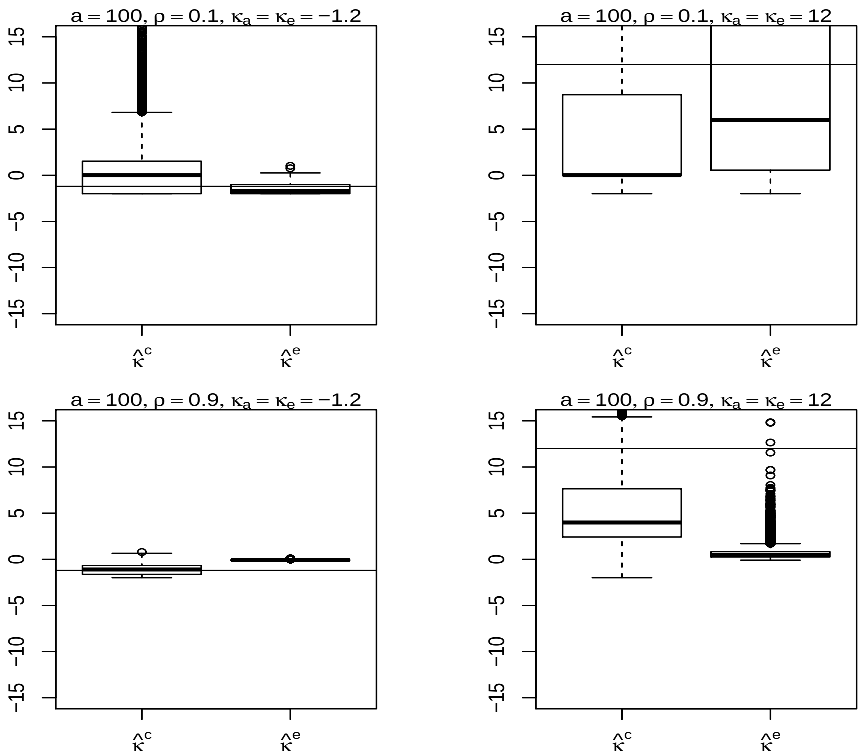
the same lines as those reported here. The estimation of κ_e is, in general, more efficient than that of κ_a.

4.3. Coverage probabilities

The coverage probabilities of the confidence intervals for ρ depend mainly on the sample size, the magnitude of ρ, the degree of imbalance, the distributions of α_i and ϵ_{ij}, the estimator of ρ and kurtosis estimation.

Figure 3 shows the coverage probabilities of the nominal 95% confidence intervals for ρ when the intraclass correlation is small ($\rho \leq .4$). The boxplots in Figure 3 represent 128 coverage probabilities corresponding to: 4 designs $\times$ 8 distributions $\times$ 4 values of ρ. It should also be mentioned that the outliers, which appear in the lower part of each plot correspond to the coverage probabilities under the Exponential(1), t(5) and $\chi^2(1)$ distributions, which have large positive kurtosis. It is apparent from Figure 3 that the

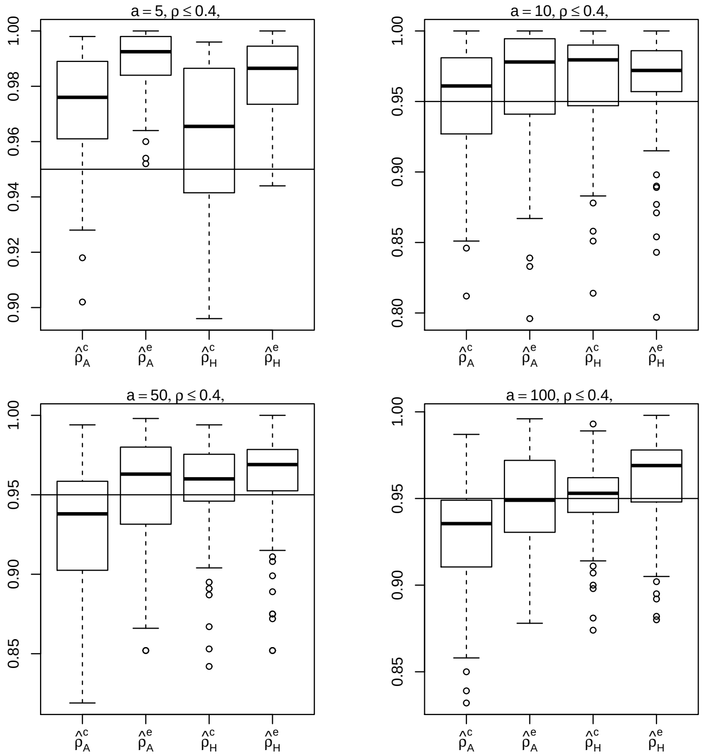
Figure 2: Boxplots for the estimators of kurtosis of α_i for U(0,1) and $\chi^2(1)$ with $\Phi = 0.26$.



intervals based on $\hat{\rho}_H^c$ were the closest to the nominal level, especially when the number of groups (a) is large. Yet, the actual coverage was well below the nominal level under leptokurtic distributions having large kurtosis.

Figure 4 shows the coverage probabilities of the nominal 95% confidence intervals for ρ when the intraclass correlation is large ($\rho \geq .5$). The boxplots in Figure 4 represent 160 coverage probabilities corresponding to: 4 designs $\times$ 8 distributions $\times$ 5 values of ρ. Here also, the outliers, which appear in the lower part of each plot correspond to the coverage probabilities under the Exponential(1), t(5) and $\chi^2(1)$ distributions, which have large positive kurtosis. It is apparent from Figure 4 that the intervals based on $\hat{\rho}_H^e$ were the closest to the nominal level. Yet again, the actual coverage was well below the nominal level under leptokurtic distributions.

Figure 3: Simulated coverage probabilities of 95% nominal confidence intervals for $\rho \leq 0.4$ with all distributions and degrees of imbalance.



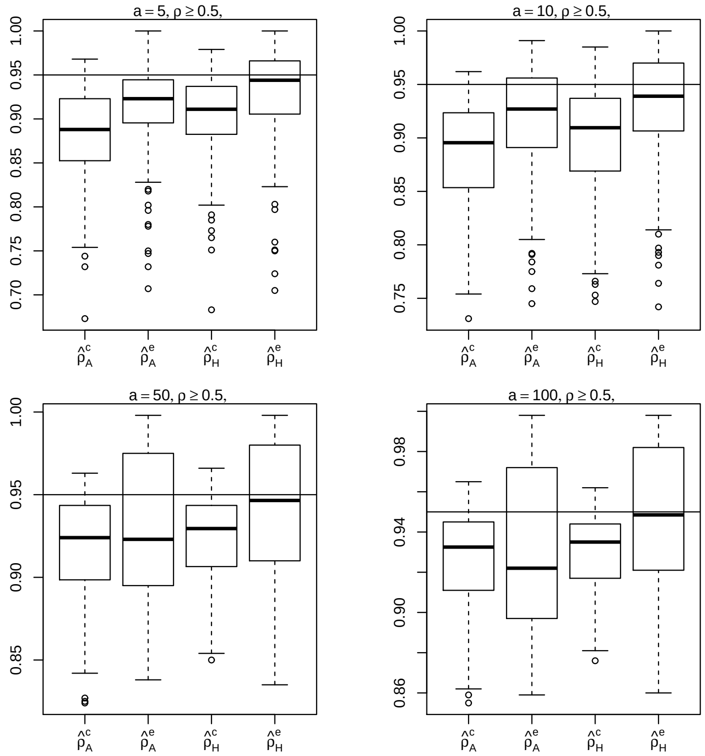
4.4. Expected lengths of the confidence intervals

Figures 5 and 6 display the ratios of average lengths of the nominal 95% confidence intervals for ρ with respect to that of the interval of Burch [2011*b*], for small and large values of ρ. For $\rho \leq .4$, it is evident from Figure 5 that the intervals based on $\hat{\rho}_H^c$ were no more than about 10% longer on the average than those based on $\hat{\rho}_A^e$. For $\rho \geq .5$, Figure 6 demonstrates that the intervals based on $\hat{\rho}_H^e$ were of about the same length as, if not narrower than, those based on $\hat{\rho}_A^e$.

5. Application: arsenic concentration in oyster tissue

Consider the example used in Burch [2011*b*] where Willie and Berman [1995] presented arsenic concentration (mg/kg) in oyster tissue samples within

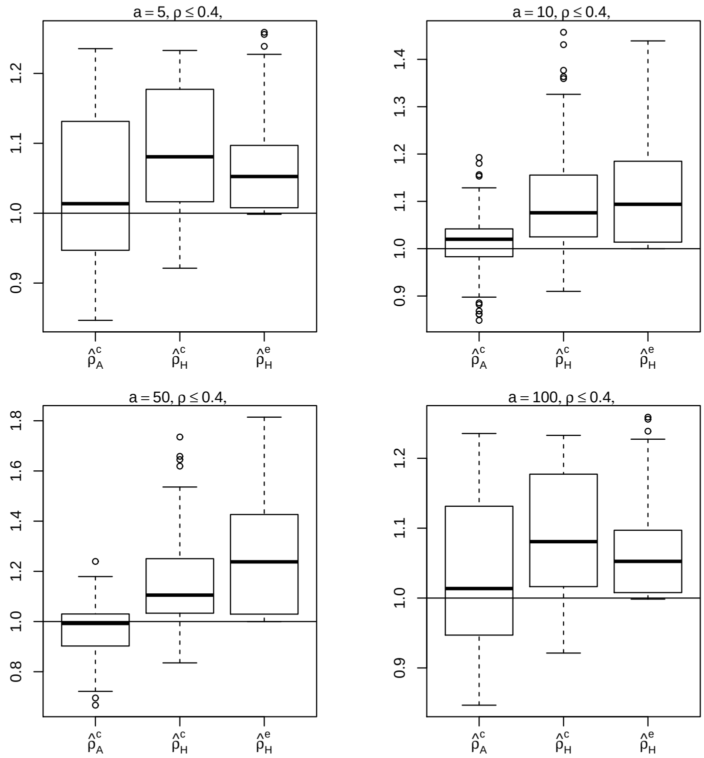
Figure 4: Simulated coverage probabilities of 95% nominal confidence intervals for $\rho \geq 0.5$ with all distributions and degrees of imbalance.



a series of studies conducted by the National Oceanic and Atmospheric Administration (NOAA) to compare inter-laboratory measurements of trace metals in marine sediments and biological tissues. An unbalanced one-way random effects model was used to estimate the intraclass correlation coefficient where $a = 31$, $n_1 = \ldots, n_{28} = 4$, $n_{29} = 2$, $n_{30} = n_{31} = 1$, and $n = \sum_{i=1}^{a} n_i = 116$. As shown by Burch [2011b], the distributions of $\hat{\alpha}_i = \bar{Y}_{i.} - \bar{Y}_{..}$ and $\hat{\epsilon}_{ij} = Y_{ij} - \bar{Y}_{i.}$, $i = 1, \ldots, 31$, $j = 1, \ldots, n_i$ are not normal and hence the conventional confidence intervals based on the normality assumption are not adequate.

In this application, one finds $\bar{\lambda} = 3.74$, $\bar{\lambda}_H = 3.26$, $\Phi = 0.87$, $\hat{\theta}_A = 9.33$ and $\hat{\theta}_H = 10.63$. The 95% confidence intervals for ρ based on each of these estimates and using the two methods of kurtosis estimation are given in Table 3. We note that the two methods of estimation give similar estimates of ρ, but the confidence intervals based $\hat{\rho}^c_H$ and $\hat{\rho}^e_H$ are shorter respectively than

Figure 5: Ratio of the expected lengths of 95% nominal confidence intervals for $\rho \leq 0.4$ with respect to those of Burch [2011*b*], for all distributions and degrees of imbalance.

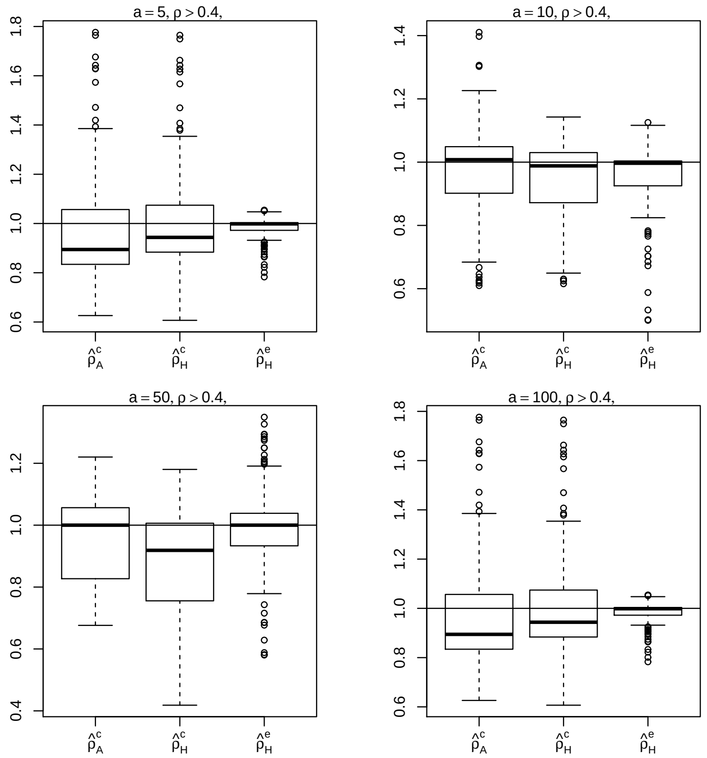


those based on $\hat{\rho}^c_A$ and $\hat{\rho}^e_A$. For this data set, the sample size and the degree of imbalance are moderate ($a = 31$, $n = 116$, $\phi = 0.87$), but the magnitudes of the kurtosis and ρ are large, thereby according to the results of the simulation study, the confidence interval based on $\hat{\rho}^e_H$ is recommended.

6. Discussion and Conclusion

In this paper, the interval estimation of the variance ratio and/or the intraclass correlation coefficient is investigated in unbalanced one-way random models under the non-normality assumptions for the distributions of the error and random effects. The confidence intervals are constructed based on two estimators; the arithmetic mean estimator $\hat{\theta}_A$ and the harmonic mean estimator $\hat{\theta}_H$. Comparing the theoretical variances of these two estimators shows that $\hat{\theta}_H$ is more efficient than $\hat{\theta}_A$, for medium and large values of ρ

Figure 6: Ratio of the expected lengths of 95% nominal confidence intervals for $\rho \geq 0.5$ with respect to those of Burch [2011*b*], for all distributions and degrees of imbalance.



($\rho > 0.4$). Since the estimated variance of $\hat{\theta}_H$ depends on the estimated kurtosis of the random factor α_i and the random errors ϵ_{ij}, we proposed a bias-corrected estimators for the kurtosis. The finite sample properties of the harmonic mean and the arithmetic mean estimators were investigated by a Monte Carlo experiment using the bias-corrected and the empirically corrected estimators of kurtosis. It turned out that the BCKEs show more stability than the ECKEs proposed by Burch [2011*b*] under different patterns of imbalance and different distributions of α_i and ϵ_{ij}.

Further, in term of coverage, the harmonic mean estimator performed well for both small and large values of ρ and for all distributions and degrees of imbalance. However, for small values of ρ, the intervals based on the harmonic mean estimator and the bias-corrected estimator of Kurtosis should be used. Otherwise, for large values of ρ, the intervals based on the harmonic mean estimator and the empirically corrected estimator of Kurtosis

Table 3: The estimates of ρ, κ_a and κ_e based on the two methods along with the Lower Bounds (LB) and Upper Bounds (UB) of the 95% confidence intervals for ρ and their lengths.

	$\hat{\rho}$	$\hat{\kappa}_a$	$\hat{\kappa}_e$	LB	UB	*length*
$\hat{\rho}_A^c$	0.90	9.92	5.20	0.70	0.97	0.28
$\hat{\rho}_A^e$	0.90	0.78	8.19	0.79	0.96	0.16
$\hat{\rho}_H^c$	0.91	7.72	5.20	0.75	0.97	0.22
$\hat{\rho}_H^e$	0.91	0.54	8.19	0.82	0.96	0.14

are recommended.

In terms of confidence interval lengths, we noticed that the expected lengths of the confidence interval based on the harmonic mean estimator with the BCKEs were a little wider than those based on the arithmetic mean estimator of Burch [2011*b*], for small values of ρ. In contrast, for large values of ρ, the expected lengths of the confidence interval based on the harmonic mean estimator with the ECKEs were of the same length as, and sometimes narrower than, than those of Burch [2011*b*].

Appendix A. Asymptotic Normality

Note that the asymptotic behavior of $\hat{\theta}_H$ is essentially determined by the ratio of two quadratic forms, namely $W_H/\mathcal{Q}_1$. Therefore, one first needs to study these quadratic forms. Mimicking the ideas of Westfall [1988] one writes both W_H and $\mathcal{Q}_1$ in terms of the vector $(\epsilon_{ij}, \alpha_i)'$ and then uses a result on the asymptotic normality of quadric forms of independent random variables such as Theorem 5.1 of Jiang [1996] or Theorem 2.1 of Bhansali et al. [2007]. It can be verified that if $0 < \mathrm{Var}(\epsilon), \mathrm{Var}(\epsilon^2) < \infty$, then $\mathcal{Q}_1$ is asymptotically normal whenever $(n-a) \to \infty$. If in addition $0 < \mathrm{Var}(\alpha), \mathrm{Var}(\alpha^2) < \infty$ then W_H is asymptotically normal whenever $a \to \infty$. However, the joint convergence of W_H and $\mathcal{Q}_1$ is what is needed for the asymptotic of $\hat{\theta}_H$. It is worth recalling that the variance of $\mathcal{Q}_1$ is essentially of the order of $(n-a)$ and that the variance of W_H is of the order of a, therefore the difference in the rates of convergence of a and n to infinity will play an important role in determining the asymptotic behavior of the vector $(W_H/(a-1), \mathcal{Q}_1/(n-a))'$. In the sequel, assume that $\sum_{i=1}^{a} n_i^k/a \to c_k$ for $k \in \{-3, -2, -1\}$. Since the n_i's are greater or equal to one, it follows that

$0 \leq c_k \leq 1$. To establish the asymptotic of $\hat{\theta}_H - \theta$ one distinguishes three cases depending on the limit of n/a.

First, if $n/a \to c_1$ where $1 < c_1 < \infty$, then $\sqrt{a}((W_H - E(W_H))/(a-1), (\mathcal{Q}_1 - E(\mathcal{Q}_1))/(n-a))'$ converges weakly to a bivariate normal with mean zero and covariance matrix Σ with entries:

$$\begin{aligned} \Sigma_{11} &= 2\sigma_a^4 + \kappa_a + 2c_{-2}\sigma_e^4 + c_{-3}\kappa_e + 4c_{-1}\sigma_a^2\sigma_e^2, \\ \Sigma_{12} &= \Sigma_{21} = \kappa_e(c_{-1} - c_{-2})/(c_1 - 1), \\ \Sigma_{22} &= 2\sigma_e^4/(c_1 - 1) + \kappa_e(c_{-1} + c_1 - 2)/(c_1 - 1)^2. \end{aligned}$$

In this case $\sqrt{a}(\hat{\theta}_H - \theta)$ converges weakly to a normal distribution with mean zero and variance $V_1 = [\Sigma_{11} + \Sigma_{22}(\theta + c_{-1})^2 - 2\Sigma_{12}(\theta + c_{-1})]/\sigma_e^4$. Note, in particular, that V_1 is just a function of θ, κ_a/σ_e^4 and κ_e/σ_e^4.

Next consider the case $n/a \to \infty$. Here the rate of convergence of W_H is much slower than that of $\mathcal{Q}_1$. One finds that $\sqrt{a}(W_H - E(W_H))/(a-1)$ converges weakly to a normal with mean zero and variance Σ_{11} and that $\sqrt{a}(\mathcal{Q}_1 - E(\mathcal{Q}_1))/(n-a)$ converges in probability to zero. It follows that $\sqrt{a}(\hat{\theta}_H - \theta)$ converges weakly to a normal distribution with mean zero and variance Σ_{11}/σ_e^4.

Finally, consider the case where $n/a \to 1$. In such case $(n-a)/a$ is going to zero and the rate of convergence of $\mathcal{Q}_1$ is much slower than that of W_H. Using Jensen's inequality one also verifies that $c_k = 1$ for $k \in \{-3, -2, -1\}$. It follows that $\sqrt{n-a}(W_H - E(W_H))/(a-1)$ converges in probability to zero and that $\sqrt{n-a}(\mathcal{Q}_1 - E(\mathcal{Q}_1))/(n-a)$ converges weakly to normal with mean zero and variance $2\sigma_e^4 + \kappa_e$. Consequently, $\sqrt{n-a}(\hat{\theta}_H - \theta)$ is asymptotically normal with mean zero and variance $(2 + \kappa_e/\sigma_e^4)(\theta + 1)^2$.

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