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Empirical processes for infinite variance autoregressive models

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Abstract

Univariate and multivariate empirical processes based on residuals of Infinite variance autoregressive processes are investigated. The results are used to develop tests of independence and Goodness of fit.

Key words: AR model, Residuals, Infinite variance, empirical process, pseudo-observations, time series.
2000 MSC: Primary 60F05, secondary 62E20.

1 Introduction

Interest in time series model with infinite variance innovations grew considerably due to the increased applications in the areas of telecommunication, finance and economics. Details and reference on these applications can be found in Adler et al. (1998) and Ling (2005). A discussion and references on financial applications can also be found in Caner (1998) and Lee and Ng (2010).

Asymptotic behavior of parameters estimates for time series models with infinite variance innovations received considerable attention in the literature. Davis and Resnick (1985a) and Davis and Resnick (1986) obtained the asymptotic distribution of least square estimator for infinite variance autoregressive (IVAR) models. Davis et al. (1992) provided asymptotic distributions for M-estimators and least absolute deviation estimators of IVAR parameters. Asymptotic behavior of parameters estimators, in the case infinite variance ARMA models (IVARMA), can be found in Davis (1996) and Mikosch et al.

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(1995). Recently a weighted least absolute deviation estimator for IVARMA parameters was considered by Pan et al. (2007). Details about rates of convergence of these estimators can be found in Section 2.

The manuscript considers diagnostic tests for errors of AR models with infinite variance innovations. It first defines and studies empirical and copula processes based of residuals of these models. Empirical process based on transformed residuals are also considered. The manuscript then uses these processes to build goodness-of-fit and randomness tests for the innovations. The paper is organized as follows. Section 2 introduces the key notations and studies the asymptotic behavior of empirical and copula processes based on residuals and squared residuals of IVAR models. Applications to goodness-of-fit and randomness tests are outlined in Section 3. Proofs are presented in Section 4.

2 Notations and main results

Consider an AR(p) model $X_i = \phi_0 + \sum_{l=1}^{p} \phi_l X_{i-l} + \varepsilon_i$, $i = 1, \ldots, n$. Assume that the errors ε's are independent and have distribution F belonging to the domain of attraction of a stable law with index $\alpha \in (0, 2)$. Let $\{e_{i,n}\}_{1 \le i \le n}$ denote the residuals that is

$$e_{i,n} = X_i - \hat{\phi}_0 - \sum_{l=1}^{p} \hat{\phi}_l X_{i-l}, \ 1 \le i \le n,$$

where $\hat{\phi} = (\hat{\phi}_0, \ldots, \hat{\phi}_p)$ is a consistent estimate of $\phi = (\phi_0, \ldots, \phi_p)$. Let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{\varepsilon_i \le t\}, \ t \in \mathbb{R}.$$

be the empirical distribution of the non-observable errors ε's and define the empirical distribution of the residuals $\tilde{F}_n$ by

$$\tilde{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{e_{i,n} \le t\}, \ t \in \mathbb{R}.$$

Let $\tilde{\mathbb{F}}_n(t) = \sqrt{n}\{\tilde{F}_n(t) - F(t)\}$, denotes the empirical process based on the residuals and let also $\mathbb{F}_n(t) = \sqrt{n}\{F_n(t) - F(t)\}$ be the empirical process for the non-observable errors ε's. Note that $\mathbb{F}_n(t) = \mathbf{B}_n(F(t))$ where

$$\mathbf{B}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\mathbf{1}\{F(\varepsilon_i) \le u\} - u\right], \ u \in [0, 1].$$

is the classical empirical process of a sequence of i.i.d uniform random variables. One generalizes these definitions to the multivariate setting in the following manner. Let m be an integer greater or equal to 1, let $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$

and define

$$\mathbf{F}_{n,m}(t) = \frac{1}{n_m}\sum_{i=1}^{n_m} \mathbf{1}\{\varepsilon_i \le t_1, \ldots, \varepsilon_{i+m-1} \le t_m\},$$

where $n_m = n - m + 1$. Note that $\mathbf{F}_{n,m}$ is the joint empirical distribution of a block of m consecutive non-observable errors ε's. Define the joint empirical distribution of the residuals $\tilde{\mathbf{F}}_{n,m}$ by

$$\tilde{\mathbf{F}}_{n,m}(\mathbf{t}) = \frac{1}{n_m}\sum_{i=1}^{n_m} \mathbf{1}\{e_{i,n} \le t_1, \ldots, e_{i+m-1,n} \le t_m\}.$$

Let $\mathbf{F}_m(\mathbf{t}) = \prod_{l=1}^{m} F(t_l)$ and let $\tilde{\mathbb{K}}_{n,m}(\mathbf{t}) = \sqrt{n_m}\{\tilde{\mathbf{F}}_{n,m}(\mathbf{t}) - \mathbf{F}_m(\mathbf{t})\}$, denotes the multivariate empirical process based on the residuals and define also $\mathbb{K}_{n,m}(\mathbf{t}) = \sqrt{n_m}\{\mathbf{F}_{n,m}(\mathbf{t}) - \mathbf{F}_m(\mathbf{t})\}$ as the multivariate empirical process for the non-observable errors ε's. Set

$$\mathbb{B}_{n,m}(u_1, \ldots, u_m) = \frac{1}{\sqrt{n_m}}\sum_{i=1}^{n_m}\left[\mathbf{1}\{F(\varepsilon_i) \le u_1, \ldots, F(\varepsilon_{i+m-1}) \le u_m\} - \prod_{l=1}^{m} u_l\right]$$

Say more about the process and relations for $0 \le u_1, \ldots, u_m \le 1$. Observe that $\mathbb{B}_{n,m}$ is the m-block empirical process of a sequence of i.i.d uniform random variables and that $\tilde{\mathbb{F}}_n(t) = \tilde{\mathbb{K}}_{n,1}(t)$.

Note that if $\phi(z) = 1 - \sum_{\ell=1}^{p} \phi_\ell z^\ell$ does not have roots with $|z| \le 1$ then the AR(p) process defined above can be represented as $X_t = \phi_0 + \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ (see for instance Brockwell and Davis (1991) page 573). The following assumptions will be needed to establish the asymptotic of the process $\tilde{\mathbb{K}}_{n,m}$.

(A.1) $P\{|\varepsilon_1| > x\} = 1 - F(x) + F(-x) = x^{-\alpha}L(x)$, where $L(x)$ is a slowly varying function at ∞.
(A.2) $\lim_{x\to\infty} P\{\varepsilon_1 > x\}/P\{|\varepsilon_1| > x\} = p$, with $0 \le p \le 1$.
(A.3) F admits a uniformly continuous bounded density f.
(A.4) $\sum_{i=1}^{\infty} |c_i|^\kappa$ converges for $\kappa < \min(\alpha, 1)$.

The asymptotic behavior of the parameters' estimator was considered by Davis and Resnick (1985b), Davis and Resnick (1986), Davis et al. (1992) and Mikosch et al. (1995). It is shown the convergence of the estimate $\hat{\phi}_l$ for $1 \le l \le p$ is in general faster then $\sqrt{n}$. In particular Davis and Resnick (1985b) and Davis and Resnick (1986) showed that the least square estimate has a rate of convergence of the order of $n^{1/\alpha}L_0(n)$ for some slowly varying function L_0 and Davis et al. (1992) showed that M-estimators have a rate of convergence a_n where $a_n = \inf\{x : nP\{|\varepsilon_1| > x\} \le 1\}$ is again of the order $n^{1/\alpha}L_1(n)$ for some slowly varying function L_1. They also showed that the least absolute deviation (LAD) estimator has a rate of convergence at least a_n and that the M-estimator of ϕ_0 is such that $\sqrt{n}\ (\hat{\phi}_0 - \phi_0)$ converges in distribution to a normal with mean zero. To establish the asymptotic of the process $\tilde{\mathbb{K}}_{n,m}$ it is therefore assumed that the estimates are such $\sqrt{n}\ (\hat{\phi}_0 - \phi_0)$ converges weakly to some Z_0 and that $\tilde{a}_n(\hat{\phi}_1 - \phi_1, \ldots, \hat{\phi}_p - \phi_p)$ converges weakly to $(Z_1, \ldots, Z_p)$ for some sequence $\tilde{a}_n = n^{1/\alpha}\tilde{L}(n)$ with $\tilde{L}$ a slowly varying function.

To state the main result let $\tilde{a}_n = n^{1/\alpha}\tilde{L}(n)$ be as indicated above and let $\hat{\Phi} = (\hat{\Phi}_0, \hat{\Phi}_1, \ldots, \hat{\Phi}_p)$ where $\hat{\Phi}_0 = \sqrt{n}\ (\hat{\phi}_0 - \phi_0)$ and $\hat{\Phi}_l = \tilde{a}_n(\hat{\phi}_l - \phi_l)$ for $l = 1, \ldots, p$.

Theorem 1 *If Assumptions (A.1)–(A.4) are satisfied, if* $(\hat{\Phi}_1, \ldots, \hat{\Phi}_p)$ *converges weakly to* $(Z_1, \ldots, Z_p)$ *and if* $(\mathbb{B}_{n,m}, \hat{\Phi}_0)$ *converges weakly to* $(\mathbb{B}_m, Z_0)$ *then* $\tilde{\mathbb{K}}_{n,m}(\mathbf{t})$ *converges weakly to the process* $\tilde{\mathbb{K}}_m$ *having the following representation* $\tilde{\mathbb{K}}_m(\mathbf{t}) = \mathbb{B}_m(F(t_1), \ldots, F(t_m)) + \sum_{j=1}^{m} Z_0 f(t_j)) \prod_{k \neq j} F(t_k)$.

All proofs are given in the Appendix. As a corollary to the above one gets the asymptotic of the univariate empirical residual process.

Corollary 2 *Under the assumptions of Theorem 1, the empirical process* $\tilde{\mathbb{F}}_n$ *converges to a continuous process* $\tilde{\mathbb{F}}$ *with representation* $\tilde{\mathbb{F}}(t) = \tilde{\mathbb{K}}_1(t) = \mathbf{B}(F(t)) + f(t)Z_0$, *where* $\mathbf{B}$ *is the classical Brownian bridge.*

An interesting feature is observed if one considers the empirical process of the squared residuals. It is shown that if f is symmetric then the asymptotic of the empirical process of the squared residuals are independent of the estimated parameters and are equivalent to the asymptotic of the empirical process based on the squares of the non-observable errors ε's. The details are given in the following Corollary.

Corollary 3 *For any fixed integer* $m \geq 1$, *let* $\tilde{\mathbb{G}}_{n,m}(\mathbf{t}) = \frac{1}{\sqrt{n_m}} \sum_{i=1}^{n_m} \mathbf{1}\{e_{i,n}^2 \leq t_1, \ldots, e_{i+m-1,n}^2 \leq t_m\} - \prod_{k=1}^{m} G(t_k)$ *where* $G(t) = F(\sqrt{t}) - F(-\sqrt{t})$ *and* $\mathbf{t} = (t_1, \ldots, t_m) \in [0, \infty)^m$. *Under the assumptions of Theorem 1, the empirical process* $\tilde{\mathbb{G}}_{n,m}$ *converges to a continuous process* $\tilde{\mathbb{G}}_m$ *with representation* $\tilde{\mathbb{G}}_m(\mathbf{t}) = \mathbb{B}_m(G(\mathbf{t})) + \sum_{j=1}^{m} \{f(\sqrt{t_j}) - f(-\sqrt{t_j})\} Z_0 \prod_{k \neq j} G(t_k)$. *If* f *is symmetric then* $\tilde{\mathbb{G}}_m(\mathbf{t}) = \mathbb{G}_m(\mathbf{t}) = \mathbb{B}_m(G(\mathbf{t}))$ *does not depend on whether the parameters are estimated or known.*

When testing for randomness of the residuals one often relies on nonparametric tests. These tests are, in general, functional of the empirical copula process whose asymptotic is provided in the next theorem.

Theorem 4 *For any fixed integer* $m \geq 1$, *let* $\tilde{\mathbb{C}}_{n,m}(\mathbf{t}) = \frac{1}{\sqrt{n_m}} \sum_{i=1}^{n_m} \mathbf{1}\{\tilde{e}_{i,n} \leq u_1, \ldots, \tilde{e}_{i+m-1,n} \leq u_m\} - \prod_{k=1}^{m} u_k$ *where* $\tilde{e}_{i,n} = \tilde{F}_n(e_{i,n})$ *denotes the normalized rank of the residual* $e_{i,n}$ *and where* $\mathbf{u} = (u_1, \ldots, u_m) \in [0,1]^m$. *Under the assumptions of Theorem 1, the empirical copula process based on residuals,* $\tilde{\mathbb{C}}_{n,m}$, *converges to the continuous process* $\mathbb{C}_m(\mathbf{u}) = \mathbb{B}_m(u_1, \ldots, u_m) - \sum_{j=1}^{m} \mathbb{B}_m(u_1, 1, \ldots, 1) \prod_{k \neq j} u_k$.

Note that the process $\mathbb{C}_m$ does not depend on whether the parameters are estimated or known. That is, critical values for tests statistics based on $\tilde{\mathbb{C}}_{n,m}$ can be easily obtained from simulation of i.i.d sequences. Details will be provided in Section 3.

3 Applications to goodness-of-fit and randomness tests

Goodness-of-fit and randomness tests are obtained by considering functional of the above mentioned processes. In what follows the section is divided into two parts; one dealing with goodness-of-fit tests and the other devoted to tests of independence.

3.1 Goodness-of-fit tests

To test the null hypothesis that states that the error distribution F is equal to a certain distribution F_0 one can use the process $\tilde{\mathbb{F}}_n$ and if F_0 is symmetric than one can use $\tilde{\mathbb{G}}_{n,m}$ instead. To be specific define

$$T_n = \int \tilde{\mathbb{F}}_n(t)^2 dF_0(t) = \int_0^1 \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n [\mathbf{1}\{F_0(e_{i,n}) \leq u\} - u] \right\}^2 du.$$

Setting $G_0(t) = F_0(\sqrt{t}) - F_0(-\sqrt{t})$ for $t \geq 0$ one then defines

$$T_n^* = \int \tilde{\mathbb{G}}_{n,1}(t)^2 dG_0(t) = \int_0^1 \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\mathbf{1}\{G_0(e_{i,n}^2) \leq u\} - u\right] \right\}^2 du.$$

It follows from Corollary 2 that T_n converges weakly to $T = \int \tilde{\mathbb{K}}_1(t)^2 dF_0(t)$. Finding critical values for T is not that obvious since the covariance function of the process $\tilde{\mathbb{K}}_1$ is in general not easy to handle. However, if the error distribution F_0, is symmetric then Corollary 3 implies that T_n^* converges weakly to $T^* = \int_0^1 \mathbb{B}(u)^2 du$ where $\mathbb{B}$ is the standard Brownian bridge. Tables for T^* are widely available; see for instance Shorack and Wellner (1986). Note that this test is equivalent to a classical goodness of fit test of a series of i.i.d random variable. In particular, when the error distribution is symmetric, one can take the series of squared residuals an apply to it any of the goodness-of-fit test (Cramer von Mises, or Kolmogorov Smirnov) the same way he would apply it a to a series of i.i.d random variables.

As mentioned in Section 2 a distribution free test of the independence of the errors can be obtained using the process $\tilde{\mathbb{C}}_{n,m}$. In particular, on could use the Cramer von Mises functional

$$\mathcal{T}_{n,m} = \int_0^1 \cdots \int_0^1 \tilde{\mathbb{C}}_{n,m}(u_1, \ldots, u_m)^2 du_1 \ldots u_m.$$

By Theorem 4, $\mathcal{T}_{n,m}$ converges weakly to $\mathcal{T}_m = \int_0^1 \cdots \int_0^1 \mathbb{C}_m(u_1, \ldots, u_m)^2 du_1 \ldots u_m$, where $\mathbb{C}_m$ is the sequential empirical copula of a sequence of i.i.d random variable. The asymptotic of $\mathcal{T}_{n,m}$ can then be obtained from the simulation of $\mathcal{T}_{n,m}$ using a sequence i.i.d uniform random variables. Alternative tests based on the copula process, developed in Genest and Rémillard (2004) and Genest et al. (2007), could used here.

4 Proofs

The process $\tilde{\mathbb{K}}_{n,m}$ is written as

$$\tilde{\mathbb{K}}_{n,m}(\mathbf{t}) = \mathbb{K}_{n,m}(\mathbf{t}) + \boldsymbol{\beta}_{n,m}(\mathbf{t}) = \mathbb{B}_{n,m}(F(t_1), \ldots, F(t_m)) + \boldsymbol{\beta}_{n,m}(\mathbf{t})$$

where

$$\boldsymbol{\beta}_{n,m}(\mathbf{t}) = \frac{1}{\sqrt{n_m}} \sum_{i=1}^{n_m} \left[\prod_{j=1}^{m} \mathbf{1}\{e_{i+j-1,n} \leq t_j\} - \prod_{j=1}^{m} \mathbf{1}\{\varepsilon_{i+j-1} \leq t_j\} \right].$$

The weak convergence of $\mathbb{B}_{n,m}$, the serial empirical process of a sequence of i.i.d. random variables, is easy to establish (see for instance Delgado (1996) or Ghoudi et al. (2001)). Therefore most of the proof resides in establishing the limiting behavior of $\boldsymbol{\beta}_{n,m}$. One notes that the process $\boldsymbol{\beta}_{n,m}$ can be written as

$$\boldsymbol{\beta}_{n,m}(\mathbf{t}) = \sum_{A \subset \mathcal{I},\ \mathrm{Card}(A) \geq 1} \boldsymbol{\beta}^A_{n,m}(\mathbf{t})$$

where $\mathcal{I} = \{1, \ldots, m\}$ and

$$\boldsymbol{\beta}^A_{n,m}(\mathbf{t}) = \frac{1}{\sqrt{n_m}} \sum_{i=1}^{n_m} \prod_{j \in A} \left[\mathbf{1}\{e_{i+j-1,n} \leq t_j\} - \mathbf{1}\{\varepsilon_{i+j-1} \leq t_j\}\right] \prod_{j \in \mathcal{I} \setminus A} \mathbf{1}\{\varepsilon_{i+j-1} \leq t_j\}.$$

To establish the asymptotic of $\boldsymbol{\beta}_{n,m}$ it will be shown that $\boldsymbol{\beta}^A_{n,m}$ is negligible for any $A \subset \mathcal{I}$ with $Card(A) \geq 2$ and that if $A = \{j\}$ then $\boldsymbol{\beta}^A_{n,m}(\mathbf{t})$ is asymptotically equivalent to $\hat{\Phi}_0 f(t_j) \prod_{k \neq j} F(t_k)$. The detailed statements are given in the next two propositions.

Proposition 5 *If Assumptions (A.1)–(A.4) are satisfied then* $\sup_{\mathbf{t}\in\mathbb{R}^m}\left|\boldsymbol{\beta}^{A}_{n,m}(\mathbf{t})\right|$ *converges to zero in probability for any subset* $A \subset \mathcal{I}$ *for which Card*$(A) \geq 2$.

Proposition 6 *Let* $j \in \mathcal{I}$, *if assumptions (A.1)–(A.4) are satisfied then*

$$\sup_{\mathbf{t}\in\mathbb{R}^m}\left|\boldsymbol{\beta}^{\{j\}}_{n,m}(\mathbf{t}) - \hat{\Phi}_0 f(t_j)\prod_{k\neq j} F(t_k)\right|$$

converges to zero in probability.

The proof of these statements shall be given after recalling some useful results and introducing some simplifying notations. First, recall that Cline (1983) showed that if F satisfies (A.1) and (A.2) then

$$\lim_{x\to\infty}\frac{P\{|X_1|>x\}}{P\{|\varepsilon_1|>x\}} = \sum_{j=0}^{\infty}|c_j|^{\alpha}, \tag{1}$$

and

$$\lim_{n\to\infty} nP\{|X_1| > a_n x\} = \sum_{j=0}^{\infty}|c_j|^{\alpha}x^{-\alpha} \quad \text{for all } x>0. \tag{2}$$

Next, for any $\Phi = (\Phi_0, \Phi_1, ...\Phi_p) \in \mathbb{R}^{p+1}$ define $\Gamma_i(\Phi) = \frac{\Phi_0}{\sqrt{n}} + \sum_{\ell=1}^{p}\frac{\Phi_\ell X_{i-\ell}}{\tilde{a}_n}$. Define $\bar{\Gamma}_i = \frac{1}{\sqrt{n}} + \sum_{\ell=1}^{p}\frac{|X_{i-\ell}|}{\tilde{a}_n}$ and for any $\mathbf{u} = (u_1,\dots,u_m) \in [0,1]^m$, and any $\delta \in \mathbb{R}$, let $\kappa^j_{n,m}(\mathbf{u},\Phi,\delta) = \gamma^j_{n,m}(\mathbf{u},\Phi,\delta) - R^j_{n,m}(\mathbf{u},\Phi,\delta) = \frac{1}{\sqrt{n_m}}\sum_{i=1}^{n_m}\kappa^j_{i,n,m}(\mathbf{u},\Phi,\delta)$ where $\kappa^j_{i,n,m}(\mathbf{u},\Phi,\delta) = \gamma^j_{i,n,m}(\mathbf{u},\Phi,\delta) - R^j_{i,n,m}(\mathbf{u},\Phi,\delta)$, and where $\gamma^j_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m}\gamma^j_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$, and $R^j_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m}R^j_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$ with

$$\gamma^j_{i,n,m}(\mathbf{u},\Phi,\delta) = \Big[\mathbf{1}\left\{\varepsilon_{i+j-1} \leq F^{-1}(u_j) + \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1}\right\} - \mathbf{1}\left\{\varepsilon_{i+j-1} \leq F^{-1}(u_j)\right\}\Big] \times \prod_{k=1,k\neq j}^{m}\mathbf{1}\{\varepsilon_{i+k-1} \leq F^{-1}(u_k)\},$$

and

$$R^j_{i,n,m}(\mathbf{u},\Phi,\delta) = \left[F\left\{F^{-1}(u_j) + \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1}\right\} - u_j\right] \times \prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1} \leq F^{-1}(u_k)\}\prod_{k=j+1}^{m}(u_k).$$

Observe that $\boldsymbol{\beta}^{\{j\}}_{n,m}(\mathbf{t}) = \gamma^j_{n,m}(\mathbf{u},\hat{\Phi},0)$ with $u_k = F(t_k)$ for $1 \leq k \leq m$. In an analogous way one defines $\kappa^{j+}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m}\kappa^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$ $\kappa^{j-}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m}\kappa^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$, where $\kappa^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta) = \gamma^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta)-$

$R^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta)$, and $\kappa^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta) = \gamma^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta) - R^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta)$, with

$$\gamma^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta) = \mathbf{1}\Big\{F^{-1}(u_j) \le \varepsilon_{i+j-1} \le F^{-1}(u_j) + \max\Big(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1}\Big)\Big\} \prod_{k=1,k\neq j}^{m} \mathbf{1}\{\varepsilon_{i+k-1} \le F^{-1}(u_k)\},$$

$$\gamma^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta) = -\mathbf{1}\Big\{F^{-1}(u_j) + \min\Big(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1}\Big) \le \varepsilon_{i+j-1} \le F^{-1}(u_j)\Big\} \prod_{k=1,k\neq j}^{m} \mathbf{1}\{\varepsilon_{i+k-1} \le F^{-1}(u_k)\},$$

$$R^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta) = \Big[F\Big\{F^{-1}(u_j) + \max(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1})\Big\} - u_j\Big] \times \prod_{k=1}^{j-1} \mathbf{1}\{\varepsilon_{i+k-1} \le F^{-1}(u_k)\} \prod_{k=j+1}^{m} (u_k)$$

and

$$R^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta) = -\Big[u_j - F\Big\{F^{-1}(u_j) + \min(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1})\Big\}\Big] \times \prod_{k=1}^{j-1} \mathbf{1}\{\varepsilon_{i+k-1} \le F^{-1}(u_k)\} \prod_{k=j+1}^{m} (u_k).$$

To complete notations one introduces $\gamma^{j+}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m} \gamma^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$, $\gamma^{j-}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m} \gamma^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$, and in a similar way $R^{j+}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m} R^{j+}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$ and $R^{j-}_{n,m}(\mathbf{u},\Phi,\delta) = \sum_{i=1}^{n_m} R^{j-}_{i,n,m}(\mathbf{u},\Phi,\delta)/\sqrt{n_m}$.

The proof of Propositions 5 and 6 shall be provided after establishing a series of technical Lemmas.

Lemma 7 *If the distribution F satisfies assumptions (A.1) and (A.2) then*

(1) $\frac{1}{\tilde{a}_n\sqrt{n}} \sum_{i=1}^{n} |X_i|$ *converges to zero in probability,*
(2) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \bar{\Gamma}_i = 1 + o_p(1)$
(3) $\sum_{i=1}^{n} \bar{\Gamma}_i^2 \mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\} \le 2 + o_p(1)$ *for any sequence of positive numbers* (ζ_n) *satisfying* $\lim_{n\to\infty} \frac{1}{\zeta_n\sqrt{n}} + n\zeta_n^{2-\alpha}\tilde{a}_n^{-\alpha}L(\zeta_n\tilde{a}_n) = 0$*, and*
(4) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}\{\bar{\Gamma}_i > \zeta_n\}$ *converges to zero in probability for any sequence of positive numbers* (ζ_n) *satisfying* $\lim_{n\to\infty} \frac{1}{\zeta_n\sqrt{n}} + \frac{n^{\frac{1}{2\alpha}+c}}{\zeta_n\tilde{a}_n} = 0$ *for some* $c > 0$.

Remark 8 *It is worth noting that statement (4) in the above Lemma applies to the case $\zeta_n = \zeta > 0$ and that both statements (3) and (4) apply if $\zeta_n = n^{-\rho}$ for $0 < \rho < \min(1/2, (2\alpha)^{-1})$.*

PROOF: Let $b_n = a_n n^\delta$ where $0 < \delta < 1/2$. One writes $\sum_{i=1}^n |X_i|/(\tilde{a}_n\sqrt{n}) = A_{n1}+A_{n2}$ where $A_{n1} = \sum_{i=1}^n |X_i|\mathbf{1}\{|X_i| \le b_n\}/(\tilde{a}_n\sqrt{n})$ and $A_{n2} = \sum_{i=1}^n |X_i|\mathbf{1}\{|X_i| > b_n\}/(\tilde{a}_n\sqrt{n})$. It follows from the stationarity of the sequence of X_i's that $E|A_{n1}| = \sqrt{n}E\{|X_1|\mathbf{1}\{|X_1| \le b_n\}\}/\tilde{a}_n$. Now for $0 < \alpha < 1$ one writes

$$E|A_{n1}| = nP\{|X_1| > a_n\}\frac{E\{|X_1|\mathbf{1}\{|X_1| \le b_n\}\}}{b_nP\{|X_1| > b_n\}}\frac{P\{|X_1| > b_n\}}{P\{|X_1| > a_n\}}\frac{b_n}{\tilde{a}_n\sqrt{n}}.$$

It is easy to see that the first term in the above right-hand-side converges to $\sum_{j=0}^\infty |c_j|^\alpha < \infty$ by (2) and (A.4), while the second term converges to $\alpha/(1-\alpha)$ by Karamata's Theorem (Feller (1971), Page 283). The third term is bounded by 1 since $b_n > a_n$ and the last term goes to zero by the choice of b_n. Therefore $E|A_{n1}|$ converges to zero and hence A_{n1} converges to zero in probability.

For $\alpha = 1$, it follows from Karamata's Theorem that $E\{|X_1|\mathbf{1}\{|X_1| \le x\}\}$ is slowly varying function, therefore for any $\eta > 0$

$$E|A_{n1}| = \frac{E\{|X_1|\mathbf{1}\{|X_1| \le b_n\}\}}{b_n^\eta}\frac{\sqrt{n}\, b_n^\eta}{\tilde{a}_n}.$$

The first term in the above right-hand-side goes to zero for any $\eta > 0$ and the second term also goes to zero for $0 < \eta < (2-\alpha)/(2+2\alpha\delta)$. This implies that $E|A_{n1}|$ converges to zero and A_{n1} converges to zero in probability.

For $1 < \alpha < 2$, $E\{|X_1|\} < \infty$ and hence $E|A_{n1}| \le \sqrt{n}\; E\{|X_1|\}/\tilde{a}_n$ which goes to zero since $\sqrt{n}\;/\tilde{a}_n \to 0$ as $\tilde{a}_n$ is regularly varying with index $1/\alpha \in (1/2, 1)$.

To complete the proof of this first statement of the Lemma observe that for any $\eta > 0$ one has $P\{|A_{n2}| > \eta\} \le P\{\cup_{i=1}^n\{|X_i| > b_n\}\} \le nP\{|X_1| > b_n\}$. But it follows from the definition of b_n that for any $M > 0$ there exist n_0 such that for $n > n_0$ one has $b_n > Ma_n$ that is $nP\{|X_1| > b_n\} \le nP\{|X_1| > Ma_n\}$ and hence by (2), $\lim_{n\to\infty} nP\{|X_1| > b_n\} \le M^{-\alpha}\sum_{j=0}^\infty |c_j|^\alpha$ for any $M > 0$. This yields $\lim_{n\to\infty} nP\{|X_1| > b_n\} = 0$ and A_{n2} converges to zero in probability.

To show statement (2) on the Lemma, it suffices to see that $\frac{1}{\sqrt{n}}\sum_{i=1}^n \bar{\Gamma}_i = 1+\frac{1}{\sqrt{n}\,\tilde{a}_n}\sum_{i=1}^n\sum_{\ell=1}^p |X_{i-\ell}|$ and that $\frac{1}{\sqrt{n}\,\tilde{a}_n}\sum_{i=1}^n\sum_{\ell=1}^p |X_{i-\ell}| \le \frac{p}{\sqrt{n}\,\tilde{a}_n}\sum_{i=1}^n |X_i| = o_p(1)$ by statement (1) of the Lemma.

For statement (3) note that

$$\sum_{i=1}^n \bar{\Gamma}_i^2\mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\} \le \sum_{i=1}^n\left(\frac{2}{n}+2\left[\sum_{\ell=1}^p\frac{|X_{i-\ell}|}{\tilde{a}_n}\right]^2\right) I\{\bar{\Gamma}_i \le \zeta_n\}$$
$$\le 2+\frac{2p}{\tilde{a}_n^2}\sum_{i=1}^n\sum_{\ell=1}^p |X_{i-\ell}|^2\mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\}.$$

By the condition on ζ_n one verifies that n large enough $1/\sqrt{n} < \zeta_n/2$ hence $|X_{i-\ell}|^2 \mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\} \le |X_{i-\ell}|^2 \mathbf{1}\{|X_{i-\ell}| \le \zeta_n \tilde{a}_n/2\}$. Therefore

$$E\left(\frac{2p}{\tilde{a}_n^2}\sum_{i=1}^{n}\sum_{\ell=1}^{p}|X_{i-\ell}|^2\mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\}\right) \le \frac{2p^2 n}{\tilde{a}_n^2}E\left(|X_1|^2\mathbf{1}\{|X_1| \le \zeta_n\tilde{a}_n/2\}\right)$$

which by Assumption (A.1), equation (1) and Karamata's Theorem is equivalent to $2^{\alpha-1}p^2\alpha\sum_{j=0}^{\infty}|c_j|^\alpha n\zeta_n^{2-\alpha}\tilde{a}_n^{-\alpha}L(\zeta_n\tilde{a}_n)/(2-\alpha)$ which goes to zero by hypothesis. One then concludes that $\frac{2p}{\tilde{a}_n^2}\sum_{i=1}^{n}\sum_{\ell=1}^{p}|X_{i-\ell}|^2\mathbf{1}\{\bar{\Gamma}_i \le \zeta_n\}$ converges in probability to zero.

To prove statement (4), recall that for n large enough $1/\sqrt{n} < \zeta_n/2$ hence $\mathbf{1}\{\bar{\Gamma}_i > \zeta_n\} \le \mathbf{1}\left\{\sum_{\ell=1}^{p}|X_{i-\ell}| > \frac{\zeta_n\tilde{a}_n}{2}\right\} \le \sum_{\ell=1}^{p}\mathbf{1}\left\{|X_{i-\ell}| > \frac{\zeta_n\tilde{a}_n}{2p}\right\}$. Therefore $E\left(\sum_{i=1}^{n}\mathbf{1}\{\bar{\Gamma}_i > \zeta_n\}/\sqrt{n}\right) \le p\sqrt{n}\ P\left\{|X_1| > \frac{\zeta_n\tilde{a}_n}{2p}\right\}$ which by (A.1) and (1) has the same limit as $(2p)^\alpha\sum_{j=0}^{\infty}|c_j|^\alpha\sqrt{n}\ (\zeta_n\tilde{a}_n)^{-\alpha}L(\zeta_n\tilde{a}_n)$. Direct algebraic manipulations show that

$$\sqrt{n}\ (\zeta_n\tilde{a}_n)^{-\alpha}L(\zeta_n\tilde{a}_n) = n^{\frac{-\alpha c}{2}}\left[\frac{\zeta_n\tilde{a}_n}{n^{\frac{1}{2\alpha}+c}}\right]^{\frac{-\alpha-\alpha^2 c}{1+2\alpha c}}\left[(\zeta_n\tilde{a}_n)^{\frac{-\alpha^2 c}{1+2\alpha c}}L(\zeta_n\tilde{a}_n)\right]$$

which goes to zero as n goes to infinity since by the conditions on the sequence (ζ_n) each of the three terms in the above right-hand-side goes zero. This implies that $\sqrt{n}\ P\left\{|X_1| > \frac{\zeta_n\tilde{a}_n}{2p}\right\}$ goes to zero and completes the proof. ■

Lemma 9 *If Δ_n converges to zero and if F satisfies (A.3) then*

$$\sup_{x,y\,:\,|F(x)-F(y)|\le\Delta_n}|f(x)-f(y)|$$

converges to zero.

Proof: It will be shown first that $\lim_{x\to\pm\infty}f(x) = 0$. We shall give the proof for $\mathbf{x}\to+\infty$ the case $\mathbf{x}\to-\infty$ being similar and is therefore omitted. We will proceed by contradiction. Assume that the there exists $\eta > 0$ and sequence $x_n \to +\infty$ such that $f(x_n) > \eta$. The uniform continuity of f implies that there exists a $\delta > 0$ such that $|f(x)-f(x_n)| \le \eta/2$ and hence $f(x) \ge \eta/2$ for all n and all $x \in (x_n-\delta, x_n+\delta)$. Next, note that $x_n \to +\infty$ implies that $1-F(x_n)$ converges to zero, that is, there exists n_0 such that for $n > n_0$ one has $1-F(x_n) \le \eta\delta/4$. But $1-F(x_n) \ge \int_{x_n}^{x_n+\delta}f(t)dt \ge \eta\delta/2$ which contradicts the above and proves that $\lim_{x\to+\infty}f(x) = 0$.

To prove the Lemma we will also proceed by contradiction and assume that there exist $\eta > 0$ and sequences x_n and y_n for which $|F(x_n)-F(y_n)| \le \Delta_n$ and $|f(x_n)-f(y_n)| > \eta$ hold for all $n > 1$. We first show that the sequences x_n and y_n must be bounded. In fact if both sequences are going to infinity then a contradiction is clear since $|f(x_n)-f(y_n)| \le |f(x_n)|+|f(y_n)|$ goes to zero by the above argument. If one sequence is going to infinity and the other one is bounded, then with out loss of generality we may assume that x_n is unbounded sequence while y_n is a bounded sequence. One can extract subsequences (x_{n_m}) and (y_{n_m}) such that $x_{n_m} \to \infty$ and $y_{n_m} \to y_0$ for some limit point y_0. The continuity of f and $|f(x_n) - f(y_n)| > \eta$ imply that $f(y_0) \geq \eta$. Also the continuity of F and $|F(x_n) - F(y_n)| \leq \Delta_n$ imply that $F(y_0) = 1$ or $F(y_0) = 0$ depending on whether $x_{n_m} \to +\infty$ or $x_{n_m} \to -\infty$. Next the uniform continuity of f implies that there exists $\delta > 0$ such that $f(t) > \eta/2$ for all $t \in (y_0 - \delta, y_0 + \delta)$. We show that this contradict the fact that $F(y_0) = 1$ or $F(y_0) = 0$. If $F(y_0) = 1$ then there is a contradiction because $0 = 1 - F(y_0) = \int_{y_0}^{\infty} f(t)dt \geq \int_{y_0}^{y_0+\delta} f(t)dt \geq \eta\delta/2$. If $F(y_0) = 0$ then the contradiction is seen because $0 = F(y_0) = \int_{-\infty}^{y_0} f(t)dt \geq \int_{y_0-\delta}^{y_0} f(t)dt \geq \eta\delta/2$. To complete the proof it just remains to show that the case of both (x_n) and (y_n) bounded also implies a contradiction. To this end assume that both sequences are bounded, that is there exist subsequences $(x_{n_m}, y_{n_m}) \to (x_0, y_0)$ for some limit point (x_0, y_0). The same arguments as before imply that $F(x_0) = F(y_0)$ and $|f(x_0) - f(y_0)| > \eta$. One immediately sees that $x_0 \neq y_0$, and can, without loss of generality, assume that $x_0 < y_0$. Observe that the continuity of the density f and $|f(x_0) - f(y_0)| > \eta$ imply that there exist $a < b$ such that $(a, b) \subset (x_0, y_0)$ and $f(t) > \eta/2$ for all $t \in (a, b)$. But a contradiction follows since $0 = F(y_0) - F(x_0) = \int_{x_0}^{y_0} f(t)dt \geq \int_a^b f(t)dt \geq \eta(b - a)/2$. ■

Lemma 10 *Let (Δ_n) be a sequence of positive numbers going to zero as n goes to infinity and let $\mathcal{B}^j(\Delta_n) = \{(\mathbf{u}, \mathbf{v}) \in [0,1]^m \times [0,1]^m : |u_j - v_j| \leq \Delta_n$ and $u_k = v_k$ for all $k \neq j\}$. If assumptions (A.1)–(A.4) are satisfied then*

$$\sup_{(\mathbf{u},\mathbf{v})\in\mathcal{B}^j(\Delta_n)} |R^{j+}_{n,m}(\mathbf{u}, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{v}, \Phi, \delta)|$$

and

$$\sup_{(\mathbf{u},\mathbf{v})\in\mathcal{B}^j(\Delta_n)} |R^{j-}_{n,m}(\mathbf{u}, \Phi, \delta) - R^{j-}_{n,m}(\mathbf{v}, \Phi, \delta)|$$

converges to zero in probability as n goes to infinity.

PROOF: The proofs for $R^{j+}_{n,m}$ and $R^{j-}_{n,m}$ are quite similar, therefore only the proof for $R^{j+}_{n,m}$ shall be given. First set $x_j = F^{-1}(u_j)$ and $x_j^* = F^{-1}(v_j)$ and let $\zeta_n = n^{-\rho}$ for some $0 < \rho < 1/4$. Observe that sequence ζ_n satisfies the conditions of Lemma 7 and note that for all $\|\Phi\| \leq B$ and for all $|\delta| \leq \delta_0$, $|R^{j+}_{n,m}(\mathbf{u}, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{v}, \Phi, \delta)|$ is bounded by

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big| F\left\{x_j + \max(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1})\right\} - F(x_j) + F(x_j^*) \\ -F\left\{x_j^* + \max(0, \Gamma_{i+j-1}(\Phi) + \delta\bar{\Gamma}_{i+j-1})\right\}\Big| \mathbf{1}\left\{(B + \delta_0)\bar{\Gamma}_{i+j-1} \leq \zeta_n\right\} \\ + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}\left\{(B + \delta_0)\bar{\Gamma}_{i+j-1} > \zeta_n\right\}.\end{aligned}$$

The second term in the above goes to zero in probability by statement (4) of Lemma 7, while applying the Mean Value Theorem shows that the first term is bounded by

$$\frac{B+\delta_0}{\sqrt{n}}\sum_{i=1}^{n}\bar{\Gamma}_{i+j-1}|f(\tilde{x}_j)-f(\tilde{x}_j^*)|\mathbf{1}\left\{(B+\delta_0)\bar{\Gamma}_{i+j-1}\le\zeta_n\right\}$$

where $\tilde{x}_j\in[x_j,x_j+0\vee(\Gamma_{i+j-1}(\Phi)+\delta\bar{\Gamma}_{i+j-1})]$ and $\tilde{x}_j^*\in[x_j^*,x_j^*+0\vee(\Gamma_{i+j-1}(\Phi)+\delta\bar{\Gamma}_{i+j-1})]$ and where $a\vee b$ denotes $\max(a,b)$. Using the triangular inequality, one bounds the above term by

$$\frac{B+\delta_0}{\sqrt{n}}\sum_{i=1}^{n}\bar{\Gamma}_{i+j-1}\Big(|f(\tilde{x}_j)-f(x_j)|+|f(\tilde{x}_j^*)-f(x_j^*)| \\ +|f(x_j)-f(x_j^*)|\Big)\mathbf{1}\left\{(B+\delta_0)\bar{\Gamma}_{i+j-1}\le\zeta_n\right\}.$$

Direct computations show that the sup of the above quantity over the set $\mathcal{B}^j(\Delta_n)$ is less or equal to

$$\left[2\sup_{|x-y|\le\zeta_n}|f(x)-f(y)|+\sup_{\substack{x,y\\|F(x)-F(y)|\le\Delta_n}}|f(x)-f(y)|\right]\frac{B+\delta_0}{\sqrt{n}}\sum_{i=1}^{n}\bar{\Gamma}_{i+j-1}.$$

The proof is then complete upon calling on the uniform continuity of f, Lemma 9, and Lemma 7. ■

Lemma 11 *Let $\bar{\mathcal{B}}^j(\eta)=\{(\mathbf{u},\mathbf{v})\in[0,1]^m\times[0,1]^m : u_j=v_j$ and $|u_k-v_k|\le\eta$ for all $k\ne j\}$. If assumptions (A.1)–(A.4) are satisfied then*

$$\sup_{(\mathbf{u},\mathbf{v})\in\bar{\mathcal{B}}^j(\eta)}|R_{n,m}^{j+}(\mathbf{u},\Phi,\delta)-R_{n,m}^{j+}(\mathbf{v},\Phi,\delta)|$$

and

$$\sup_{(\mathbf{u},\mathbf{v})\in\bar{\mathcal{B}}^j(\eta)}|R_{n,m}^{j-}(\mathbf{u},\Phi,\delta)-R_{n,m}^{j-}(\mathbf{v},\Phi,\delta)|$$

converges to zero in probability as η goes to zero.

Proof: Again, only the proof for $R_{n,m}^{j+}$ shall be given. It will be shown that the convergence is uniform for all $\|\Phi\|\le B$ and $|\delta|\le\delta_0$. Recall that $u_j=v_j$ and observe that

$$
\begin{aligned}
&|R_{n,m}^{j+}(\mathbf{u},\Phi,\delta)-R_{n,m}^{j+}(\mathbf{v},\Phi,\delta)| \\
&=\frac{1}{\sqrt{n_m}}\Big|\sum_{i=1}^{n_m}\Big[F\{F^{-1}(u_j)+\max(0,\Gamma_{i+j-1}(\Phi)+\delta\bar{\Gamma}_{i+j-1})\}-u_j\Big] \\
&\quad\Big[\prod_{k=j+1}^{m}u_k\prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)\}-\prod_{k=j+1}^{m}v_k\prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(v_k)\}\Big]\Big| \\
&\le\frac{1}{\sqrt{n_m}}\sum_{i=1}^{n_m}\Big|F\{F^{-1}(u_j)+\max(0,\Gamma_{i+j-1}(\Phi)+\delta\bar{\Gamma}_{i+j-1})\}-u_j\Big| \\
&\quad\Big|\prod_{k=j+1}^{m}u_k\prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)\}-\prod_{k=j+1}^{m}v_k\prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(v_k)\}\Big|
\end{aligned}
$$

Since F is uniformly continuous and admits a bounded density one easily verifies that $|R_{n,m}^{j+}(\mathbf{u},\Phi,\delta)-R_{n,m}^{j+}(\mathbf{v},\Phi,\delta)|\le(A_1+A_2)\sqrt{n/n_m}$ where

$$
A_1=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{\bar{\Gamma}_i>\zeta_n\}
$$

and

$$
\begin{aligned}
A_2=&\frac{\|f\|(B+\delta_0)}{\sqrt{n}}\sum_{i=1}^{n_m}\bar{\Gamma}_{i+j-1}\Big|\Big(\prod_{k=j+1}^{m}u_k-\prod_{k=j+1}^{m}v_k\Big)\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\} \\
&+\prod_{k=j+1}^{m}v_k\Big(\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\}-\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le v_k\}\Big)\Big|\mathbf{1}\{\bar{\Gamma}_{i+j-1}\le\zeta_n\},
\end{aligned}
$$

with $\zeta_n=n^{-\rho}$ for $0<\rho<1/4$. By Lemma 7, the term A_1 goes to zero in probability, the proof will then follows if one shows that A_2 goes to zero in probability. This is achieved by first letting $C=\|f\|(B+\delta_0)$ and noting that

$$
\begin{aligned}
A_2\le&\frac{C}{\sqrt{n}}\Big|\prod_{k=j+1}^{m}u_k-\prod_{k=j+1}^{m}v_k\Big|\sum_{i=1}^{n_n}\bar{\Gamma}_{i+j-1}+\frac{C}{\sqrt{n}}\sum_{i=1}^{n_m}\bar{\Gamma}_{i+j-1}\mathbf{1}\{\bar{\Gamma}_{i+j-1}\le\zeta_n\} \\
&\Big[\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\vee v_k\}-\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\wedge v_k\}\Big] \\
\le&\frac{m\eta C}{\sqrt{n}}\sum_{i=1}^{n}\bar{\Gamma}_i+\frac{C}{\sqrt{n}}\Big(\sum_{i=1}^{n}\bar{\Gamma}_i^2\mathbf{1}\{\bar{\Gamma}_i\le\zeta_n\}\Big)^{\frac{1}{2}} \\
&\Big(\sum_{i=1}^{n_m}\Big[\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\vee v_k\}-\prod_{k=1}^{j-1}\mathbf{1}\{F(\varepsilon_{i+k-1})\le u_k\wedge v_k\}\Big]\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Upon calling on Lemma 7 one bounds the above by

$$m\eta C(1+o_p(1))+C(2+o_p(1))\left\{2\|\mathbf{F}_{n,m}-\mathbf{F}_m\|^{\frac{1}{2}}+\Big(\prod_{k=1}^{j-1}u_k\vee v_k-\prod_{k=1}^{j-1}u_k\wedge v_k\Big)^{1/2}\right\}$$

which is smaller or equal to $m\eta C(1+o_p(1))+C(2+o_p(1))\sqrt{m\eta}+2C(2+o_p(1))\|\mathbf{F}_{n,m}-\mathbf{F}_m\|^{\frac{1}{2}}$. By the adaptation of Glivenco-Cantelli Theorem, the latter goes to zero in probability as n goes to infinity and η goes to zero. ■

Lemma 12 *If assumptions (A.1)–(A.4) are satisfied then, for any compact subset* $\mathbf{D}$ *of* $\mathbb{R}^{p+1}$ *and any* $\delta_0>0$,

$$\sup_{|\delta|\le\delta_0}\sup_{\mathbf{u}\in[0,1]^m}\sup_{\Phi\in\mathbf{D}}|\kappa^{j+}_{n,m}(\mathbf{u},\Phi,\delta)|$$

and

$$\sup_{|\delta|\le\delta_0}\sup_{\mathbf{u}\in[0,1]^m}\sup_{\Phi\in\mathbf{D}}|\kappa^{j-}_{n,m}(\mathbf{u},\Phi,\delta)|$$

converge to zero in probability.

Proof: The two terms being quite similar, therefore only the proof for $\kappa^{j+}_{n,m}$ is presented. First, for $\eta>0$ and let $D_1,\dots,D_K$ be a finite cover of $\mathbf{D}$ with diameter smaller than η and centers denoted $\Phi^1,\dots,\Phi^K$ respectively. Let also $\tilde\delta_1,\dots,\tilde\delta_M$ be a partition of $[-\delta_0,\delta_0]$ with mesh between η and 2η. Note that $\Phi\in\mathbf{D}$ implies that $\Phi\in D_k$ for some $1\le k\le K$ and $|\delta|\le\delta_0$ implies that $\tilde\delta_r<\delta\le\tilde\delta_{r+1}$ for some $1\le r\le M-1$. Set $\tilde\delta_0=-\delta_0-\eta$ and $\tilde\delta_{M+1}=\delta_0+\eta$ it then follows from the definition of $\kappa^{j+}_{n,m}$ that

$$\kappa^{j+}_{n,m}(\mathbf{u},\Phi,\delta)\ge\kappa^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r-1})-R^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r+2})+R^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r-1})$$

$$\kappa^{j+}_{n,m}(\mathbf{u},\Phi,\delta)\le\kappa^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r+2})+R^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r+2})-R^{j+}_{n,m}(\mathbf{u},\Phi^k,\tilde\delta_{r-1}).$$

It is quite easy to verify that, for any $\mathbf{u}\in[0,1]^m$ and any Φ,

$$|R^{j+}_{n,m}(\mathbf{u},\Phi,\tilde\delta_{r+2})-R^{j+}_{n,m}(\mathbf{u},\Phi,\tilde\delta_{r-1})|\le 6\eta\|f\|\frac{1}{\sqrt{n_m}}\sum_{i=1}^n\bar\Gamma_i$$

which by Lemma 7 and (A.3) goes to zero in probability when η goes to zero. Therefore the result will be proven if one shows that $\sup_{\mathbf{u}\in[0,1]^m}|\kappa^{j+}_{n,m}(\mathbf{u},\Phi,\delta)|$ converges to zero in probability for any fixed $\Phi\in\mathbf{D}$ and any finite δ. To achieve this, choose (K_n) as a sequence of positive integers satisfying $\lim_{n\to\infty}\sqrt{n}\,/K_n+K_n/n^{1-\rho}=0$, for some $0<\rho<1/2$, and let $\Delta_n=1/K_n$. Choose $0=b_0<b_1<\cdots<b_{K_n}=1$ to be a partition of $[0,1]$ with mesh less or equal to Δ_n. Let K be a positive integer and set $a_k=k/K$ for $k=0,1,\dots,K$ note that $a_0,a_1,\dots,a_K$ form a partition of $[0,1]$. One sees that for any $\mathbf{u}\in(0,1)^m$ there exist $0\le r_j\le K_n-1$ such $b_{r_j}<u_j\le b_{r_j+1}$ and $0\le r_k\le K-1$ such $a_{r_k}<u_k\le a_{r_k+1}$ for $1\le k\le m$. Note that the partition $b_0,\dots,b_{K_n}$ will only

used for the jth component while the partition $a_0, \ldots, a_K$ will be used for the rest of the component. Setting

$$\mathbf{u}^+ = (a_{r_1+1}, \ldots, a_{r_{j-1}+1}, b_{r_j+1}, a_{r_{j+1}+1}, \ldots, a_{r_m+1}),$$

$$\mathbf{u}^{*+} = (a_{r_1+1}, \ldots, a_{r_{j-1}+1}, b_{r_j}, a_{r_{j+1}+1}, \ldots, a_{r_m+1}),$$
$$\mathbf{u}^- = (a_{r_1}, \ldots, a_{r_{j-1}}, b_{r_j}, a_{r_{j+1}}, \ldots, a_{r_m}),$$

and

$$\mathbf{u}^{*-} = (a_{r_1}, \ldots, a_{r_{j-1}}, b_{r_j+1}, a_{r_{j+1}}, \ldots, a_{r_m}),$$

one obtains

$$\gamma^{j+}_{i,n,m}(\mathbf{u}, \Phi, \delta) \le \gamma^{j+}_{i,n,m}(\mathbf{u}+, \Phi, \delta) + \mathbf{1}\{b_{r_j} \le u_{i+j-1} \le b_{r_j+1}\} \prod_{\substack{k=1\\k\neq j}}^{m} \mathbf{1}\{u_{i+k-1} \le a_{r_k+1}\},$$

$$\gamma^{j+}_{i,n,m}(\mathbf{u}, \Phi, \delta) \ge \gamma^{j+}_{i,n,m}(\mathbf{u}-, \Phi, \delta) - \mathbf{1}\{b_{r_j} \le u_{i+j-1} \le b_{r_j+1}\} \prod_{\substack{k=1\\k\neq j}}^{m} \mathbf{1}\{u_{i+k-1} \le a_{r_k}\},$$

$$R^{j+}_{i,n,m}(\mathbf{u}, \Phi, \delta) \le R^{j+}_{i,n,m}(\mathbf{u}+, \Phi, \delta) + \{b_{r_j+1} - b_{r_j}\} \prod_{k=1}^{j-1} \mathbf{1}\{u_{i+k-1} \le a_{r_k+1}\} \prod_{k=j+1}^{m} a_{r_k+1},$$

and

$$R^{j+}_{i,n,m}(\mathbf{u}, \Phi, \delta) \ge R^{j+}_{i,n,m}(\mathbf{u}-, \Phi, \delta) - \{b_{r_j+1} - b_{r_j}\} \prod_{k=1}^{j-1} \mathbf{1}\{u_{i+k-1} \le a_{r_k}\} \prod_{k=j+1}^{m} a_{r_k},$$

Combining these facts, one easily obtains $\kappa^{j+}_{n,m}(\mathbf{u}, \Phi, \delta) \ge \kappa^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta) - (R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)) - (\mathbb{B}_{n,m}(\mathbf{u}^{*-}) - \mathbb{B}_{n,m}(\mathbf{u}^-)) - 2\Delta_n\sqrt{n}$ and $\kappa^{j+}_{n,m}(\mathbf{u}, \Phi, \delta) \le \kappa^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) + (R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)) + (\mathbb{B}_{n,m}(\mathbf{u}^+) - \mathbb{B}_{n,m}(\mathbf{u}^{*+})) + 2\Delta_n\sqrt{n}$, hence $|\kappa^{j+}_{n,m}(\mathbf{u}, \Phi, \delta)|$ is less or equal to

$$|\kappa^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta)| + |\kappa^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)| + |R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)|$$
$$+ |\mathbb{B}_{n,m}(\mathbf{u}^+) - \mathbb{B}_{n,m}n(\mathbf{u}^{*+})| + |\mathbb{B}_{n,m}(\mathbf{u}^{*-}) - \mathbb{B}_{n,m}(\mathbf{u}^-)| + 2\Delta_n\sqrt{n}\ .$$

The term $2\Delta_n\sqrt{n} = 2\sqrt{n}\ /K_n$ converges to zero by the choice of K_n. Since $\|u^+ - u^{*+}\|$ and $\|u^{*-} - u^-\|$ are smaller than Δ_n which goes to zero, then $\sup_{\mathbf{u}\in[0,1]^m} |\mathbb{B}_{n,m}(\mathbf{u}^+) - \mathbb{B}_{n,m}(\mathbf{u}^{*+})|$ and $\sup_{\mathbf{u}\in[0,1]^m} |\mathbb{B}_{n,m}(\mathbf{u}^{*-}) - \mathbb{B}_{n,m}(\mathbf{u}^-)|$ both converge to zero in probability by the tightness of the empirical process $\mathbb{B}_{n,m}$. While, by the triangular inequality, $|R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)| \le |R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^{*+}, \Phi, \delta)| + |R^{j+}_{n,m}(\mathbf{u}^{*+}, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)|$. It follows from Lemmas 10 and 11 that $\sup_{\mathbf{u}\in[0,1]^m} |R^{j+}_{n,m}(\mathbf{u}^+, \Phi, \delta) - R^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)|$ goes to zero in probability as η goes to zero. The proof will then be complete if one shows that

$$\max_{0\le r_j\le K_n} \max_{0\le r_k\le K; k\neq j} |\kappa^{j+}_{n,m}(\mathbf{u}^-, \Phi, \delta)|$$

converges to zero in probability.

Let ϵ be an arbitrary positive number and note that

$$P\left\{\max_{0\le r_j\le K_n}\max_{0\le r_k\le K;k\ne j}|\kappa_{n,m}^{j+}(\mathbf{u}^-,\Phi,\delta)|>\epsilon\right\}$$
$$\le K_nK^{m-1}\max_{0\le r_j\le K_n}\max_{0\le r_k\le K;k\ne j}P\left\{|\kappa_{n,m}^{j+}(\mathbf{u}^-,\Phi,\delta)|>\epsilon\right\}.$$

Now for $h\in\{1,2,\ldots,m\}$ define

$$\kappa_{n,m}^{j+,h}(u,\Phi,\delta)=\frac{1}{\sqrt{n_m}}\sum_{k=1}^{\lfloor n/m\rfloor}\kappa_{(k-1)m+h,n,m}^{j+}(u,\Phi,\delta).$$

Since $|\kappa_{i,n,m}^{j+}(u,\Phi,\delta)|\le 1$, one verifies that $|\kappa_{n,m}^{j+}(u,\Phi,\delta)-\sum_{h=1}^m\kappa_{n,m}^{j+,h}(u,\Phi,\delta)|\le m/\sqrt{n_m}\le\epsilon/2$ for n large enough. It follows that $P\left\{|\kappa_{n,m}^{j+}(\mathbf{u}^-,\Phi,\delta)|>\epsilon\right\}\le\sum_{h=1}^m P\left\{|\kappa_{n,m}^{j+,h}(\mathbf{u}^-,\Phi,\delta)|>\epsilon/(2m)\right\}$. For each $h\in\{1,2,\ldots,m\}$ one notices that $\kappa_{n,m}^{j+,h}(u^-,\Phi,\delta)$ is a sum of martingale differences with respect to the filtration $\mathcal{F}_k^{j,h}=\sigma\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{km+h+j-2}\}$. Applying Rosenthal's inequality (Hall and Heyde (1980)) shows that there exists a constant C such that

$$P\left\{|\kappa_{n,m}^{j+,h}(u^-,\Phi,\delta)|>\epsilon\right\}\le\frac{C}{n_m^2\epsilon^4}\sum_{k=1}^{\lfloor n/m\rfloor}E\left[\left(\kappa_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta)\right)^4\right]$$
$$+\frac{C}{n_m^2\epsilon^4}E\left\{\sum_{k=1}^{\lfloor n/m\rfloor}E\left[\left(\kappa_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta)\right)^2\Big|\mathcal{F}_{k-1}^h\right]\right\}^2.$$

Since $|\kappa_{n,m}^{j+,h}(u,\Phi,\delta)|\le 1$ the first term in the above inequality is bounded $\frac{C}{n_m\eta^4}$. For the second term observe that

$$E\left\{\sum_{k=1}^{\lfloor n/m\rfloor}E\left[\left(\kappa_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta)\right)^2\Big|\mathcal{F}_{k-1}^h\right]\right\}^2$$
$$=E\left\{\sum_{k=1}^{\lfloor n/m\rfloor}R_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta)(1-R_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta))\right\}^2$$
$$\le E\left\{\sum_{k=1}^{\lfloor n/m\rfloor}R_{(k-1)*m+h,n,m}^{j+}(u^-,\Phi,\delta)\right\}^2$$
$$\le E\Bigg[\sum_{k=1}^{\lfloor n/m\rfloor}F\Big\{F^{-1}(u_j)+\max\Big(0,\Gamma_{(k-1)*m+h+j-1}(\Phi)+\delta\bar\Gamma_{(k-1)*m+h+j-1}\Big)\Big\}-u_j\Bigg]^2$$
$$\le E\Bigg[\sum_{k=1}^{\lfloor n/m\rfloor}F\Big\{F^{-1}(u_j)+B\bar\Gamma_{(k-1)*m+h+j-1}\Big\}-u_j\Bigg]^2,$$

where $B = \|\Phi\| + |\delta|$. One can easily check that the above is bounded by the sum of

$$2E\bigg\{\sum_{k=1}^{\lfloor n/m \rfloor}\Big[F\Big\{F^{-1}(u_j) + B\bar{\Gamma}_{(k-1)*m+h+j-1}\Big\} - u_j\Big] \times \mathbf{1}\{\max_{1\le\ell\le p}|X_{(k-1)*m+h+j-1-\ell}| \le a_n\}\bigg\}^2 \quad (3)$$

and

$$2E\bigg\{\sum_{k=1}^{\lfloor n/m \rfloor}\Big[F\Big\{F^{-1}(u_j) + B\bar{\Gamma}_{(k-1)*m+h+j-1}\Big\} - u_j\Big] \times \mathbf{1}\{\max_{1\le\ell\le p}|X_{(k-1)*m+h+j-1-\ell}| > a_n\}\bigg\}^2. \quad (4)$$

Applying the mean value theorem shows that (3) is bounded by

$$\|f\|^2B^2\Big[4n + 4E\Big(\sum_{i=1}^{n}\sum_{l=1}^{p}\frac{|X_{i-l}|}{\tilde{a}_n}\mathbf{1}\{|X_{i-l}| \le a_n\}\Big)^2\Big]$$

which is also bounded by

$$\|f\|^2B^2\Big[4n + 4n^2p^2E\Big(\frac{|X_1|^2}{\tilde{a}_n^2}\mathbf{1}\{|X_1| \le a_n\}\Big)\Big].$$

Simple computations show that (4) is bounded

$$2E\Big(\sum_{i=1}^{n}\mathbf{1}\{\max_{1\le l\le p}|X_{i-l}| > a_n\}\Big)^2 \le 2n^2pP\{|X_1| > a_n\}.$$

By assumptions, (2) and Feller (1971) (Theorem 2 p 283), one has both $nP\{|X_1| > a_n\}$ and $nE\Big(\frac{|X_1|^2}{a_n^2}\mathbf{1}\{|X_1| \le a_n\}\Big)$ converge to finite limits. Combining these facts yields

$$P\left\{\max_{0\le r_j\le K_n}\max_{0\le r_k\le K;k\ne j}|\kappa_{n,m}^{j+}(\mathbf{u}^-, \Phi, \delta)| > \epsilon\right\} \le \frac{C_2K_nK^{m-1}}{n\epsilon^4}\left(1 + \frac{a_n^2}{\tilde{a}_n^2}\right)$$

for some constant C_2. The above converges to zero for any $\epsilon > 0$ by the choice of K_n and the fact that $a_n/\tilde{a}_n$ is a slowly varying function of n. ■

Lemma 13 *If assumptions (A.1)–(A.4) are satisfied then for any compact set* $\mathbf{D} \subset \mathbb{R}^{p+1}$,

$$\sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\left|R_{n,m}^j(\mathbf{u}, \Phi, 0) - \Phi_0 f(F^{-1}(u_j))\prod_{k=1;k\ne j}^{m}u_k\right|$$

converges to zero in probability.

Proof: One first writes

$$\sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\left|R^j_{n,m}(\mathbf{u},\Phi,0)-\Phi_0 f(F^{-1}(u_j))\prod_{k=1;k\neq j}^m u_k\right|$$
$$\leq \sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\left|R^j_{n,m}(\mathbf{u},\Phi,0)-R^j_{n,m}(\mathbf{u},\Phi^*,0)\right|$$
$$+\sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\left|R^j_{n,m}(\mathbf{u},\Phi^*,0)-\Phi_0 f(F^{-1}(u_j))\prod_{k=1;k\neq j}^m u_k\right|$$

where $\Phi^*=(\Phi_0,0,\ldots,0)$. Applying the Mean Value Theorem one sees that

$$\left|R^j_{n,m}(\mathbf{u},\Phi^*,0)-\Phi_0 f(F^{-1}(u_j))\prod_{k=1;k\neq j}^m u_k\right|$$
$$\leq \frac{|\Phi_0|f(F^{-1}(u_j))\prod_{k=j+1}^m u_k}{n_m}\left|\sum_{i=1}^{n_m}\Big[\prod_{k=1}^{j-1}\mathbf{1}\{\varepsilon_{i+k-1}\leq F^{-1}(u_k)\}-\prod_{k=1}^{j-1}u_k\Big]\right|$$
$$+|\Phi_0|\left|f(\xi)-f(F^{-1}(u_j))\right|$$

where $\xi\in[F^{-1}(u_j),F^{-1}(u_j)+\Phi_0/\sqrt{n}\,]$. Straightforward computations yields

$$\sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\left|R^j_{n,m}(\mathbf{u},\Phi^*,0)-\Phi_0 f(F^{-1}(u_j))\prod_{k=1;k\neq j}^m u_k\right|$$
$$\leq B\|f\|\sup_{\mathbf{t}\in\mathbb{R}^m}|\mathbf{F}_{n,m}(\mathbf{t})-\mathbf{F}_m(\mathbf{t})|+B\sup_{|x-y|\leq B/\sqrt{n}}|f(x)-f(y)|$$

where $B=\sup_{\Phi\in\mathbf{D}}\|\Phi\|$. The above goes to zero by the uniform continuity of f and an adaptation of the Glivenko-Cantelli Lemma. To complete the proof note that

$$\sup_{\Phi\in D}\sup_{\mathbf{u}\in[0,1]^m}\left|R^j_{n,m}(\mathbf{u},\Phi,0)-R^j_{n,m}(\mathbf{u},\Phi^*,0)\right|\leq B\|f\|\frac{1}{\sqrt{n_m}}\sum_{i=1}^{n_m}\sum_{\ell=1}^{p}\frac{|X_{i+j-1-\ell}|}{\tilde{a}_n}$$
$$\leq\frac{Bp\|f\|\sqrt{n}}{\sqrt{n_m}}\frac{1}{\sqrt{n}\,\tilde{a}_n}\sum_{i=1}^{n}|X_i|.$$

which goes to in probability to zero by Lemma 7 and the fact that m is finite.

4.1 Proof of Proposition 5:

The proof follows similar arguments to those used in Ghoudi and Rémillard (2004). First define

$$\gamma_{n,m}^{A}(\mathbf{u},\Phi,\delta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\prod_{j\in A}\Big(\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)+\Gamma_{i+j-1}(\Phi)+\delta\bar{\Gamma}_{i+j-1}\}$$
$$-\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)\}\Big)\prod_{j\in A^c}\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)\}$$

and observe that $\boldsymbol{\beta}_{n,m}^{A}(\mathbf{t}) = \gamma_{n,m}^{A}(\mathbf{u},\hat{\Phi},0)$ where $u_k = F^{-1}(t_k)$ for $1\le k\le m$. Since $\hat{\Phi}$ is tight, to prove Proposition 5 it suffices to show that

$$\sup_{\Phi\in\mathbf{D}}\sup_{\mathbf{u}\in[0,1]^m}\gamma_{n,m}^{A}(\mathbf{u},\Phi,0)$$

converges to zero in probability for any compact subset $\mathbf{D}\subset\mathbb{R}^{p+1}$. To do so assume that $\mathbf{D}$ is a compact subset of $\mathbb{R}^{p+1}$ and set $B=\sup_{\Phi\in D}\|\Phi\|$. Let ζ be a positive number and note that $|\gamma_{n,m}^{A}(\mathbf{u},\Phi,0)|\le S_1+S_2$ where

$$S_1 = \Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\prod_{j\in A}\Big(\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)+\Gamma_{i+j-1}(\Phi)\}$$
$$-\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)\}\Big)\prod_{j\in A^c}\mathbf{1}\{\varepsilon_{i+j-1}\le F^{-1}(u_j)\}\mathbf{1}\{\max_{1\le j\le m}\bar{\Gamma}_{i+j-1}\le\frac{\zeta}{B}\}\Big|$$

and $S_2 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{\max_{1\le j\le m}\bar{\Gamma}_{i+j-1}>\frac{\zeta}{B}\}$. One verifies that the term $S_2\le\sum_{j=1}^{m}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{\bar{\Gamma}_{i+j-1}>\frac{\zeta}{B}\}$ and hence converges to zero in probability for any $\zeta>0$ by Lemma 7. To consider S_1 note that since $Card(A)\ge 2$ there exist $j,j_0\in A$ such that $1\le j<j_0\le m$. Recall that $\Gamma_k(\Phi)\le B\bar{\Gamma}_k$ for any $\Phi\in\mathbf{D}$ and observe that

$$\begin{aligned}S_1 \le{}& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{F^{-1}(u_j)<\varepsilon_{i+j-1}\le F^{-1}(u_j)+B\bar{\Gamma}_{i+j-1}\}\\
&\times\mathbf{1}\{F^{-1}(u_{j_0})-\zeta<\varepsilon_{i+j_0-1}\le F^{-1}(u_{j_0})+\zeta\}\\
&\times\prod_{k\in A\setminus\{j,j_0\}}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)+\zeta\}\prod_{k\in A^c}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)\}\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}\{F^{-1}(u_j)-B\bar{\Gamma}_{i+j-1}<\varepsilon_{i+j-1}\le F^{-1}(u_j)\}\\
&\times\mathbf{1}\{F^{-1}(u_{j_0})-\zeta<\varepsilon_{i+j_0-1}\le F^{-1}(u_{j_0})+\zeta\}\\
&\times\prod_{k\in A\setminus\{j,j_0\}}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)+\zeta\}\prod_{k\in A^c}\mathbf{1}\{\varepsilon_{i+k-1}\le F^{-1}(u_k)\}\end{aligned}$$

The right-hand-side of the above is equal to $\gamma_{n,m}^{j+}(\mathbf{u}^{\zeta},0,B)-\gamma_{n,m}^{j+}(\mathbf{u}^{*\zeta},0,B)-\gamma_{n,m}^{j-}(\mathbf{u}^{\zeta},0,-B)+\gamma_{n,m}^{j-}(\mathbf{u}^{*\zeta},0,-B)$ where $\mathbf{u}^{\zeta}$ is such that $u_k^{\zeta}=F(F^{-1}(u_k)+\zeta)$ for $k\in A\setminus\{j\}$ and $u_k^{\zeta}=u_k$ otherwise and $\mathbf{u}^{*\zeta}$ is such that $u_k^{*\zeta}=u_k^{\zeta}$ for all $k\ne j_0$ and $u_{j_0}^{*\zeta}=F(F^{-1}(u_{j_0})-\zeta)$. Adding and subtracting $R_{n,m}^{j+}$ or $R_{n,m}^{j-}$ as needed one obtains $S_1\le|\kappa_{n,m}^{j+}(\mathbf{u}^{\zeta},0,B)|+|\kappa_{n,m}^{j+}(\mathbf{u}^{*\zeta},0,B)|+|\kappa_{n,m}^{j-}(\mathbf{u}^{\zeta},0,-B)|+$

$|\kappa^{j-}_{n,m}(\mathbf{u}^{*\zeta},0,-B)| + |R^{j+}_{n,m}(\mathbf{u}^{\zeta},0,B) - R^{j+}_{n,m}(\mathbf{u}^{*\zeta},0,B)| + |R^{j-}_{n,m}(\mathbf{u}^{\zeta},0,-B) - R^{j-}_{n,m}(\mathbf{u}^{*\zeta},0,-B)|$. Next note that $\mathbf{u}^{\zeta}$ and $\mathbf{u}^{*\zeta}$ only differ in their j_0th component and that $|u^{\zeta}_{j_0} - u^{*\zeta}_{j_0}| \leq 2\|f\|\zeta$. Since ζ can be made arbitrarily small, calling on Lemmas 11 and 12 completes the proof. ■

4.2 Proof of Proposition 6:

As noted earlier $\boldsymbol{\beta}^{\{j\}}_{n,m}(\mathbf{t}) = \gamma^{j}_{n,m}(\mathbf{u},\hat{\Phi},0) = \kappa^{j}_{n,m}(\mathbf{u},\hat{\Phi},0) + R^{j}_{n,m}(\mathbf{u},\hat{\Phi},0)$ for $\mathbf{u} = (u_1,\ldots,u_m)$ with $u_k = F(t_k)$ for $1 \leq k \leq m$. It follows from Lemma 12 and the tightness of $(\hat{\Phi})$ that $\sup_{\mathbf{u}\in[0,1]^m} |\kappa^{j}_{n,m}(\mathbf{u},\hat{\Phi},0)|$ converges to zero in probability. To complete the proof note that the tightness of $(\hat{\Phi})$ and Lemma 13 imply that $\sup_{\mathbf{u}\in[0,1]^m} |R^{j}_{n,m}(\mathbf{u},\hat{\Phi},0) - \hat{\Phi}_0 f(F^{-1}(u_j)) \prod_{k\neq j} u_k|$ converges to zero in probability and hence $\sup_{\mathbf{t}\in\mathbb{R}^m} |\boldsymbol{\beta}^{\{j\}}_{n,m}(\mathbf{t}) - \hat{\Phi}_0 f(t_j)) \prod_{k\neq j} F(t_k)|$ converges to zero in probability. ■

4.3 Proof of Theorem 1:

By definition $\tilde{\mathbb{K}}_{n,m}(\mathbf{t}) = \mathbb{B}_{n,m}(F(t_1),\ldots,F(t_m)) + \boldsymbol{\beta}_{n,m}(\mathbf{t})$. Since $\boldsymbol{\beta}_{n,m}(\mathbf{t}) = \sum_{j=1}^{m} \boldsymbol{\beta}^{\{j\}}_{n,m}(\mathbf{t}) + \sum_{A\subset\mathcal{I};Card(A)>1} \boldsymbol{\beta}^{A}_{n,m}(\mathbf{t})$, it follows from Propositions 6 and 5 that $\sup_{\mathbf{t}\in\mathbb{R}^m} |\boldsymbol{\beta}_{n,m}(\mathbf{t}) - \sum_{j=1}^{m} \hat{\Phi}_0 f(t_j)) \prod_{k\neq j} F(t_k)|$ converges to zero in probability. Therefore $\tilde{\mathbb{K}}_{n,m}(\mathbf{t})$ is asymptotically equivalent to $\mathbb{B}_{n,m}(F(t_1),\ldots,F(t_m)) + \sum_{j=1}^{m} \hat{\Phi}_0 f(t_j)) \prod_{k\neq j} F(t_k)$ which, by assumptions, converges weakly to the process $\mathbb{B}_m(F(t_1),\ldots,F(t_m)) + \sum_{j=1}^{m} Z_0 f(t_j)) \prod_{k\neq j} F(t_k)$. ■

4.4 Proof of Theorem 4:

Let $\tilde{F}_n^{-1}(u) = \inf\{t \in \mathbf{R} : \tilde{F}_n(t) \geq u\}$ for $0 < u < 1$ and note that $|[F\{\tilde{F}_n^{-1}(u)\} - u] + [\tilde{F}_n\{\tilde{F}_n^{-1}(u)\} - F\{\tilde{F}_n^{-1}(u)\}]| = |\tilde{F}_n\{\tilde{F}_n^{-1}(u)\} - u| \leq 1/n$. Next, by definition $\mathbb{C}_{n,m}(u_1,\ldots,u_m)$ is equal to $\tilde{\mathbb{K}}_{n,m}(\tilde{F}_n^{-1}(u_1),\ldots,\tilde{F}_n^{-1}(u_m)) + \sqrt{n_m}[\prod_{i=1}^{m} F\{\tilde{F}_n^{-1}(u_i)\} - \prod_{i=1}^{m} u_i]$. Direct algebraic manipulations show that

$$\sup_{\mathbf{u}\in[0,1]^m} \sqrt{n_m} \left| \prod_{i=1}^{m} F\{\tilde{F}_n^{-1}(u_i)\} - \prod_{i=1}^{m} u_i - \sum_{j=1}^{m} \left[F\{\tilde{F}_n^{-1}(u_j)\} - \tilde{F}_n\{\tilde{F}_n^{-1}(u_j)\}\right] \prod_{i\neq j} u_i \right|$$

converges to zero in probability. Using the above and Theorem 1, one concludes that $\mathbb{C}_{n,m}(u_1,\ldots,u_m)$ converges to

$$\tilde{\mathbb{K}}_m(F^{-1}(u_1),\ldots,F^{-1}(u_m)) - \sum_{j=1}^{m} \tilde{\mathbb{K}}_m(F^{-1}(u_j),\infty,\ldots,\infty) \prod_{i\neq j} u_i$$

which simplifies to $\mathbb{B}_m(u_1,\ldots,u_m) - \sum_{j=1}^m \mathbb{B}_m(u_j,1,\ldots,1)\prod_{i\neq j} u_i$.

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